Mathematics 111

Spring 2011

Homework 7

1. (#3 p. 852) Show that the degree one representations of a finite group G are in bijective correspondence with the degree one representations of the abelian group G/G', where G' is the commutator subgroup of G.

2. (#15, p 853) Exhibit all one-dimensional complex representations of a finite cyclic group, and determine which are inequivalent.

3. (#16, p 853) Exhibit all one-dimensional complex representations of a finite abelian group. Deduce that the number of inequivalent degree one complex representations of a finite abelian group equals the order of the group. Hint: First decompose the abelian group into a direct sum of cyclic groups and use the previous problem.

4. #20, p 854) Prove that the number of degree one complex representations of any finite group G equals the index $[G : G']$ where G' is the commutator subgroup.

5. (based upon #19, p 854) The goal is to prove that any finite-dimensional complex representation of a finite abelian group is the direct sum of one-dimensional (necessarily irreducible) representations without the use of Maschke's theorem. In the process, we prove a number of interesting results in linear algebra.

   Let k be a field and V an n-dimensional vector space over k. The first goal is to prove that if $\mathfrak{S}$ is a set of commuting, diagonalizable linear operators in $GL(V)$, then there is a basis for V with respect to which every element of $\mathfrak{S}$ has a diagonal matrix representation, that is the elements of $\mathfrak{S}$ are simultaneously diagonalizable.

   (a) Consider an element $T \in \mathfrak{S}$. We know that T is diagonalizable iff its minimal polynomial m_T is the product of distinct linear factors: $m_T = (x-\lambda_1)\cdots(x-\lambda_r)$. In our consideration of the Jordan form we wrote $V = V(x-\lambda_1)\oplus\cdots\oplus V(x-\lambda_r)$, where $V(x-\lambda_i) = \{v \in V \mid (T-\lambda_i I)^k(v) = 0 \text{ for some } k \geq 1\}$. Show that $V(x-\lambda_i)$ is the λ_i-eigenspace E_{λ_i}.

   (b) To proceed you may want to consider a proof by induction on n. Let $T \in \mathfrak{S}$ and write $V = \oplus_\lambda E_\lambda$. Explain why it is possible (wlog) to choose T so that there are at least two eigenspaces E_λ.

   (c) Show that for all $S \in \mathfrak{S}$, S is invariant on each E_λ. Show that S is diagonalizable on each E_λ.

   (d) Show that any set of elements in $\mathfrak{S}$ can be simultaneously diagonalized.

   (e) Explain why whether $\mathfrak{S}$ is a finite or infinite set doesn't matter.

   (f) Finally show that any finite-dimensional complex representation of a finite abelian group is the direct sum of one-dimensional (necessarily irreducible) representations without the use of Maschke's theorem.