# Mathematics 111

Spring 2011

Homework 6

1. Let $T$ be a linear operator on a finite dimensional vector space $V$ over a field $k$, and let $q_1 \mid q_2 \mid \cdots \mid q_s$ be the invariant factors associated to $V$ as a torsion $k[x]$-module. Show that $q_1 q_2 \cdots q_s = c_T$, where $c_T$ is the characteristic polynomial of $T$. Show that the minimal polynomial of $T$ divides the characteristic polynomial. Note that this proves the Cayley-Hamilton theorem. *Hint:* You may use without proof (though you should think about it) that the determinant of a matrix of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ with $A$ and $C$ square matrices is the product $\det(A)\det(C)$.

2. Find all rational and Jordan canonical forms of a matrix in $M_5(\mathbb{C})$ having minimal polynomial $x^2(x-1)$. Be sure to give the corresponding invariants and the characteristic polynomials.

3. Show that any linear operator $T$ on a finite dimensional vector space (over a field of characteristic not equal to 2) which satisfies $T^2 = I$ is diagonalizable. Give all possible Jordan forms for $4 \times 4$ matrices $A$ with $A^2 = I$.

4. Consider a matrix of the form $A = \begin{pmatrix} \lambda & \mu & 0 & \dots & 0 \\ 0 & \lambda & \mu & \dots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \lambda & \mu \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$, i.e., with diagonal $\lambda$ and $\mu$ on the superdiagonal. Find the Jordan canonical form(s) of $A$. The answer should depend slightly on $\mu$. What does your answer say about the form of Jordan blocks as introduced in the text in comparison with the way we defined them?

5. #24, p501. Prove that there are no $3 \times 3$ $A$ matrices over $\mathbb{Q}$ which satisfy $A^8 = I$, but $A^4 \neq I$.

6. #19, p501. Prove that all $n \times n$ matrices over a field $F$ having a fixed characteristic polynomial $f \in F[x]$ are similar if and only if $f$ factors into distinct irreducibles in $F[x]$.