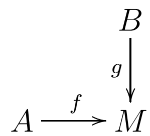
Mathematics 111

Spring 2011

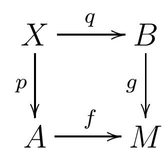
Homework 3

1. (Commutative diagrams gone mad) Given a ring R with identity and R-modules A, B, M, consider the following diagram with R-linear maps f, g:

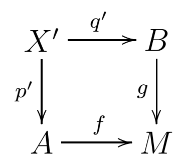


A *pullback* for this diagram (also called a fiber product of f and g) consists of the following data:

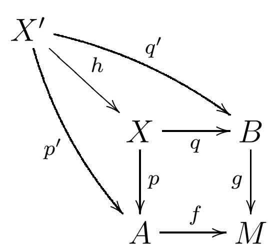
- A module X and R-linear maps $p : X \to A$, $q : X \to B$ making the following diagram commute.



- For every R-module X' admitting a commutative diagram of linear maps (same f, g)



there is a unique R-linear map $h : X' \to X$ such that the following diagram commutes:



Now to the exercise: Let $X = \{(a, b) \in A \times B \mid f(a) = g(b)\}$, p and q the standard projections to the factors A and B. Show that X together with the associated data form a pullback, i.e., verify that X is an R-module and that the above universal mapping property holds for this choice of X and maps p, q.

Remark: Dummit and Foote introduce the pullback as a subset of the direct sum $A \oplus B$. Categorically, you should find the direct product a more useful perspective.

2. Let R be a ring, and consider two exact sequences of R-modules

$$0 \longrightarrow K \longrightarrow P \xrightarrow{\varphi} M \longrightarrow 0 \qquad 0 \longrightarrow K' \longrightarrow P' \xrightarrow{\varphi'} M \longrightarrow 0$$

where P and P' are projective. Show that as R-modules $K' \oplus P \cong K \oplus P'$.

Hint: Show there is an exact sequence

$$0 \longrightarrow \ker \pi \longrightarrow X \xrightarrow{\pi} P \longrightarrow 0$$

with $\ker \pi \cong K'$ and where X is the fiber product of φ and φ' as in the first problem. From this deduce that $X \cong K' \oplus P$. Similarly, show $X \cong K \oplus P'$.

3. Let V be a vector space over a field K, and let $\varphi : V \to V$ be a K-linear map.

   (a) If V is finite dimensional, show that there is a positive integer m so that $Im(\varphi^m) \cap Ker(\varphi^m) = \{0\}$.

   (b) Show by example that this need not hold if V is infinite dimensional.

4. Let V be an n-dimensional vector space over a field K, and $T : V \to V$ a K-linear map satisfying $T^n = 0$, but $T^{n-1} \neq 0$.

   (a) Show that for $1 \leq j \leq n$, $rank(T^j) := \dim_k(Im(T^j))$ equals $n - j$.

   (b) Using the preceding part, show that there is a basis of V so that the matrix of T with respect to this basis is strictly upper triangular, which is to say it has zeros on and below the main diagonal.

5. Let $T : V \to V$ be a linear operator on a complex vector space. For two different T, we view V as a $\mathbb{C}[x]$-module in the usual way. In each case determine whether or not V is decomposable (as a $\mathbb{C}[x]$-module) . If it is, write V as an appropriate direct sum and express the matrix of T with respect to the associated basis. If V is indecomposable, prove it.

   (a) First consider T whose matrix with respect to a basis $B = \{e_1, e_2, e_3\}$ is $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.

   (b) Second, consider (a different) T whose matrix with respect to a basis $B = \{e_1, e_2, e_3\}$ is $\begin{pmatrix} 3 & 0 & 5 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.