Mathematics 111

Spring 2009

Homework 5

1. Let R be a PID and M a finitely generated R-module. Show that M is projective if and only if it is free.

2. Let M be a submodule of $\mathbb{Z}^n$ having group index p in M, i.e., $[\mathbb{Z}^n : M] = p$, where p is a prime. Show that M is free of rank n, and there is a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{Z}^n$ so that $M = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{n-1} \oplus \mathbb{Z}pe_n$.

3. Show that a vector $v = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ extends to a basis $\{v, v_2, \ldots, v_n\}$ of $\mathbb{Z}^n$ if and only if the a_i are coprime, that is $a_1\mathbb{Z} + \cdots + a_n\mathbb{Z} = \mathbb{Z}$. *Hint:* For one direction, come up with a short exact sequence that splits.

4. Let F_1 and F_2 be free modules of (not necessarily the same) finite rank over a PID R. Let $\varphi : F_1 \to F_2$ be R-linear and nontrivial. Show that there exists bases $\{v_1, \ldots, v_n\}$ of F_1 and $\{w_1, \ldots, w_m\}$ of F_2 together with elements $a_1, \ldots, a_r$ of R so that

$$\varphi(v_i) = \begin{cases} a_i w_i & 1 \leq i \leq r \\ 0 & r+1 \leq i \leq n, \end{cases}$$

with $a_1 \mid a_2 \mid \cdots \mid a_r$, and the ideals $a_j R$ uniquely determined.

5. Let $A = \begin{pmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{pmatrix}$.

   (a) If $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^2$ is a $\mathbb{Z}$-linear map whose matrix with respect to the standard bases is A, determine the structure of the cokernel $\mathbb{Z}^2/Im(\varphi)$ as a direct sum of cyclic groups. Find a minimal set of generators for the quotient. *Hint:* The image of φ is the span of the columns (i.e., the column space), and you may assume without loss of generality that elementary column operations (over $\mathbb{Z}$) leave the column space unchanged. Explain how your answer is connected to the invariant factor theorem.

   (b) Determine all integer solutions to $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$. *Hint:* Elementary row operations (over $\mathbb{Z}$) do not change the kernel.