Mathematics 111

Spring 2007

Homework 5

1. Show that a vector $v = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ extends to a basis $\{v, v_2, \ldots, v_n\}$ of $\mathbb{Z}^n$ if and only if the a_i are coprime, that is $a_1\mathbb{Z} + \cdots + a_n\mathbb{Z} = \mathbb{Z}$. Hint: For one direction, come up with a short exact sequence that splits.

2. Let $A = \begin{pmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{pmatrix}$.

   (a) If $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^2$ is a $\mathbb{Z}$-linear map whose matrix with respect to the standard bases is A, determine the structure of the cokernel $\mathbb{Z}^2/Im(\varphi)$ as a direct sum of cyclic groups. Find a minimal set of generators for this quotient Hint: The image of φ is the span of the columns (i.e., the column space), and you may assume wlog that elementary column operations (over $\mathbb{Z}$) leave the column space unchanged.

   Explain how your answer is connected to the elementary divisors theorem.

   (b) Determine all integer solutions to $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hint: Elementary row operations (over $\mathbb{Z}$) do not change the kernel.