Mathematics 111

Spring 2007

Homework 3

1. (Pullbacks) Given a ring A with identity and A-modules M, M', M'', consider the following diagram with A-linear maps f, g:

$$\begin{array}{ccc} & & M'' \\ & & \big\downarrow g \\ M' & \xrightarrow{f} & M \end{array}$$

A *pullback* for this diagram (also called a fiber product of f and g) consists of the following data:

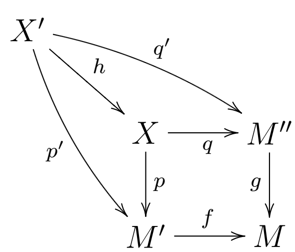
(a) An object X and A-linear maps $p : X \to M'$, $q : X \to M''$ making the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{q} & M'' \\ p\big\downarrow & & \big\downarrow g \\ M' & \xrightarrow{f} & M \end{array}$$

(b) For every commutative diagram of linear maps (same f, g)

$$\begin{array}{ccc} X' & \xrightarrow{q'} & M'' \\ p'\big\downarrow & & \big\downarrow g \\ M' & \xrightarrow{f} & M \end{array}$$

there is a unique A-linear map $h : X' \to X$ such that the following diagram commutes:



Let $X = \{(m', m'') \in M' \times M'' \mid f(m') = g(m'')\}$, p and q the standard projections to the factors M' and M''. Show that X together with the associated data form a pullback, i.e., verify that X is an A-module and that the universal mapping property (b) holds for this choice of X and maps p, q.

2. Let A be a ring, and consider two exact sequences of A-modules

$$0 \longrightarrow K \longrightarrow P \xrightarrow{\varphi} M \longrightarrow 0 \qquad 0 \longrightarrow K' \longrightarrow P' \xrightarrow{\varphi'} M \longrightarrow 0$$

where P and P' are projective. Show that as A-modules $P \oplus K' \cong P' \oplus K$. *Hint:* Show there is an exact sequence

$$0 \longrightarrow \ker \pi \longrightarrow X \xrightarrow{\pi} P \longrightarrow 0$$

with $\ker \pi \cong K'$ and where X is the fiber product of φ and φ' as in the first problem. From this deduce that $X \cong P \oplus K'$. Similarly, show $X \cong P' \oplus K$.