Math 108. Topics in Combinatorics.

Problem Set 4. Due on Thursday, 11/10/16.

1. How many SYT of shape (n^n) have main diagonal $(1, 4, 9, 16, \ldots, n^2)$?

2. Let $f^{\lambda/2}$ denote the number of SYT of shape λ having the entry 2 in the first row. Evaluate the sums
$$\sum_{\lambda \vdash n} f^{\lambda/2} f^{\lambda} \qquad \text{and} \qquad \sum_{\lambda \vdash n} \left(f^{\lambda/2}\right)^2.$$

3. Consider a hexagon with equal angles and integer side lengths given counter-clockwise by r, s, t, r, s, t. How many equilateral rhombi (consisting of two equilateral triangles of side 1) are needed to tile the hexagon?

4. Let Γ be a region on a square grid, and let τ be a tiling of Γ by dominoes.
   (a) Prove that the parity of the number of vertical dominoes in τ only depends on Γ, not on the particular tiling τ.
   (b) Prove that if Γ is simply connected (it has no "holes"), then every two tilings τ and τ' are connected by a series of 2×2 flips (one such flip switches two adjacent vertical dominoes with two horizontal ones).

5. (*) Let $f(n)$ be the number of partitions of $2n$ whose Young diagram can be tiled with n dominoes. For instance, (4,3,3,3,1) is such a partition. Prove that $f(n)$ is equal to the number of ordered pairs (λ, μ) of partitions satisfying $|\lambda| + |\mu| = n$.