Math 108. Topics in Combinatorics.
Problem Set 3. Due on Thursday, 10/27/16.

1. Fix $t \geq 0$. Show that $p_{n-t}(n)$ becomes eventually constant as $n \to \infty$. What is this constant c_t? What is the least value of n for which $p_{n-t}(n) = c_t$?

2. Prove that
$$\prod_{i\geq 0}(1+q^{2i+1}) = \sum_{k\geq 0}\frac{q^{k^2}}{(1-q^2)\cdots(1-q^{2k})},$$

3. Prove the following statements:
   (a) The number of partitions of n into parts congruent to ± 1 mod 3 equals the number of partitions of n where every part appears at most twice.
   (b) The number of partitions of n into parts congruent to ± 1 mod 6 equals the number of partitions of n into distinct parts congruent to ± 1 mod 3.

4. Let $f(n)$ (respectively, $g(n)$) be the number of partitions $\lambda \vdash n$ into distinct parts, such that the largest part λ_1 is even (respectively, odd). Prove that
$$f(n) - g(n) = \begin{cases} 1, & \text{if } n = k(3k+1)/2 \text{ for some } k \geq 0 \\ -1, & \text{if } n = k(3k-1)/2 \text{ for some } k \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

5. Let M be a random $n \times n$ matrix with entries in the finite field $\mathbb{F}_q$, where each entry is chosen uniformly and independently at random. Show that with probability at least $1/4$, M is non-singular (i.e., it has nonzero determinant).
Hint: The generating function for pentagonal numbers may be surprisingly useful here.

6. Find a bijective proof (using a Franklin-type involution, like in the proof of Euler's Pentagonal Theorem) of Jacobi's identity in the form:
$$\prod_{n=1}^{\infty}(1-x^n y^{n-1})(1-x^{n-1}y^n)(1-x^n y^n) = 1 + \sum_{n=1}^{\infty}(-1)^n\left(x^{\frac{n(n+1)}{2}}y^{\frac{n(n-1)}{2}} + x^{\frac{n(n-1)}{2}}y^{\frac{n(n+1)}{2}}\right).$$