Your Name Here

Math 105

Homework 4

1. (3-3-12) Let K be a number field, and $\mathcal{O}_K^\times$ its unit group. Show that $\varepsilon \in \mathcal{O}_K^\times$ iff $N_{K/\mathbb{Q}}(\varepsilon) = \pm 1$.

2. (3-3-12) Let $D \in \mathbb{Z}$, squarefee $D < 0$, and set $K = \mathbb{Q}(\sqrt{D})$. Let $i = \sqrt{-1}$ and ω a primitive cube root of unity in $\mathbb{C}$. Show that the units of K form a finite cyclic group. In particular, show

$$\mathcal{O}_K^\times = \begin{cases} \langle i \rangle = \{\pm 1, \pm i\} & \text{if } D = -1, \\ \langle -\omega \rangle = \{\pm 1, \pm \omega, \pm \omega^2\} & \text{if } D = -3, \\ < -1 >= \{\pm 1\} & \text{otherwise.} \end{cases}$$

3. (3-3-12) Let $D \in \mathbb{Z}$, squarefee $D > 1$, and set $K = \mathbb{Q}(\sqrt{D})$.
   (a) Show that $\mathcal{O}_K^\times \cong \{\pm 1\} \times \langle \varepsilon \rangle$ where ε has infinite order.
   (b) The unique generator ε which satisfies $\varepsilon > 1$ is called the fundamental unit of K. Show that the fundamental unit of $K = \mathbb{Q}(\sqrt{2})$ is $\varepsilon = 1 + \sqrt{2}$. Note: it is not always so easy: The fundamental unit of $\mathbb{Q}(\sqrt{94})$ is $\varepsilon = 2143295 + 221064\sqrt{94}$.