Your Name Here

Math 105

Homework 2

1. (3-1-7) Universal Mapping Property for completions Let L be a field complete with respect to a valuation $|\cdot|_L$ and K a field with valuation $|\cdot|_K$, and $\widehat{K}$ its completion under the valuation. Assume that there is an isometry $\lambda : K \to L$. Show there is a unique isometric field embedding $\rho : \widehat{K} \to L$ making the diagram below commute.

$$\begin{tikzcd} \widetilde{K} \arrow[r, "\rho"] & L \\ K \arrow[u, hook] \arrow[ur, "\lambda"'] & \end{tikzcd} \tag{1}$$

2. (3-1-18) Let K be a field with valuation $|\cdot|_K$. Show that $|\cdot|_K$ is non-archimedean iff there exists $C > 0$ with $|m|_K \leq C$ for all $m \in \mathbb{Z}$.

3. We showed in class that a number field K embeds as a discrete subring of its adele ring, $\mathbb{A}_K$.

   (a) Show that $\mathbb{A}_K$ has a Hausdorff topology, so that it is a locally compact topological ring with a Hausdorff topology.

   (b) Show the K embeds as a closed subset of $\mathbb{A}_K$.

4. Let G be a topological group, i.e., G has a topology with respect to which the group operations are continuous.

   (a) Show that every open subgroup of G is closed, and every closed subgroup of G of finite index is open.

   (b) Show that every subgroup $H \leq G$ which contains a neighborhood of the identity is open.

   (c) Show that if H is s a subgroup of G, the coset space G/H is discrete if and only if H is open.