Your Name Here

Math 105

Homework 1

1. Let K be a number field of degree n over $\mathbb{Q}$, q a prime in $\mathbb{Z}$, and suppose that $q\mathcal{O}_K = Q_1^{e_1} \cdots Q_r^{e_r}$ is the factorization of $q\mathcal{O}_K$ as a product of prime ideals in $\mathcal{O}_K$. Show that
$$[\mathcal{O}_K : q\mathcal{O}_K] = q^n = \prod_{i=1}^{r} [\mathcal{O}_K : Q_i]^{e_i}.$$
Note that this proves that $n = [K : \mathbb{Q}] = \sum_{i=1}^{r} e_i f_i$ where $[\mathcal{O}_K : Q_i] = q^{f_i}$.

2. Let K be a number field, $[K : \mathbb{Q}] = n$ and let $\alpha \in \mathcal{O}_K$ with $K = \mathbb{Q}(\alpha)$. It is not always the case that the ring of integers of a number field has the form $\mathbb{Z}[\alpha]$, but we can often show that it is by gaining information about how large $\mathbb{Z}[\alpha]$ is as a subring of $\mathcal{O}_K$.

   Suppose that $[\mathcal{O}_K : \mathbb{Z}[\alpha]] = m$. Show that $\Delta(1, \alpha, \ldots, \alpha^{n-1}) = m^2 d_K$ where d_K is the discriminant of K.

3. Let L/K be an extension of number fields with $n = [L : K]$. Suppose that $L = K(\alpha)$, and put $f = m_{\alpha,K}$ be its minimal polynomial. Show that
$$\Delta(1, \alpha, \ldots, \alpha^{n-1}) = (-1)^{\frac{n(n-1)}{2}} N_{L/K}(f'(\alpha)),$$
where $N_{L/K}(\alpha) = \prod_{\sigma \in E} \sigma(\alpha)$ is the norm from L to K and f' is the formal derivative of f; here E is the set of all embedding of L into an algebraic closure of K which fix K pointwise.

4. Let K be a number field with $K = \mathbb{Q}(\alpha)$ for some $\alpha \in \mathcal{O}_K$, and suppose that $[\mathcal{O}_K : \mathbb{Z}[\alpha]] = m$. Let $f = m_{\alpha,\mathbb{Q}} = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ be the minimal polynomial of α. Show that if f is Eisenstein with respect to some rational prime p ($p \mid a_i$ for all i, but $p^2 \nmid a_0$), then $p \nmid m$.

   Some hints.

   - First show that the Eisenstein condition implies that $\alpha^n/p \in \mathcal{O}_K$ and that $p^2 \nmid N_{K/\mathbb{Q}}(\alpha)$.
   - Next assume that $p \mid m = [\mathcal{O}_K : \mathbb{Z}[\alpha]]$. We shall try to obtain a contradiction. To begin, show there exists $\xi = \mathcal{O}_K \setminus \mathbb{Z}[\alpha]$ with $p\xi = b_0 + b_1\alpha + \cdots + b_{n-1}\alpha^{n-1} \in \mathbb{Z}[\alpha]$, and not all b_i divisible by p.
   - Let j be the smallest index with $p \nmid b_j$ and put $\eta = (b_j\alpha^j + \cdots + b_{n-1}\alpha^{n-1})p^{-1}$. Show that η and $\zeta = b_j\alpha^{n-1}p^{-1}$ are elements of $\mathcal{O}_K$.
   - Explore $N_{K/\mathbb{Q}}(p\zeta)$ to produce the contradiction.

(continued on reverse)

5. Let $K = \mathbb{Q}(\sqrt[3]{5})$. The goal is to prove that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{5}]$ using some of the tools you have developed in the problems above.

   (a) Show that the discriminant $\Delta(1, \sqrt[3]{5}, \sqrt[3]{25}) = -3^3 \cdot 5^2$. Note that this implies an upper bound on the index $[\mathcal{O}_K : \mathbb{Z}[\sqrt[3]{5}]]$.

   (b) Note that $f = x^3 - 5$ is Eisenstein with respect to the prime 5, and $g = (x+5)^3 - 5$ is Eisenstein with respect to the prime 3. How can you use this to reach the desired conclusion?

6. Let p be a prime in $\mathbb{Z}$, ζ a primitive p^nth root of unity in $\mathbb{C}$, and put $K = \mathbb{Q}(\zeta)$. Show that $\mathcal{O}_K = \mathbb{Z}[\zeta]$.

   Some hints.

   - Show that the discriminant $d_K = (-1)^{p(p-1)/2} p^N$ where $N = n\phi(p^n) - p^{n-1}$ where ϕ is the Euler totient function. Recall that the minimal polynomial of ζ is the p^nth cyclotomic polynomial which for prime powers is easy to compute.
   - Then show that the element $(1 - \zeta)$ has minimal polynomial which is Eisenstein with respect to the prime p.