Supplementary homework problems, due April 20, 2009

1. Let $\mathcal{P}$ be a set of prime numbers. Prove that

$$\sum_{p\in\mathcal{P}} \frac{1}{p} = \infty \text{ if and only if } \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p}\right) = 0.$$

2. Prove that if $\mathcal{P}$ is a set of primes with an infinite reciprocal sum, then the set of integers not divisible by any member of $\mathcal{P}$ has asymptotic density 0.

3. Let $\mathcal{S}$ be the set of integers which can be expressed as the sum of two coprime squares, so that $\mathcal{S} = \{1, 2, 5, 10, 13, 17, 25, \ldots\}$. Look up (or recall) the theorem which classifies which integers are in $\mathcal{S}$ and use this together with the results in Section 2 of Chapter 4 and the exercise above to show that $\mathcal{S}$ has asymptotic density 0.