Math 105: Introduction to Modular Forms
Homework 1: Due Friday, April 18

(1) Prove that $\Lambda(\omega_1, \omega_2) = \Lambda(\omega_1', \omega_2')$ if and only if there are integers a, b, c and d with $ad - bc = \pm 1$ such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

(2) Let $\Lambda = \Lambda(\omega_1, \omega_2)$ and let $\wp = \wp_\Lambda$.

(a) For any $c \in \mathbb{C}$, prove that $\wp(z) - c$ has exactly two zeros (or a single double zero) in $\Pi' = \{a\omega_1 + b\omega_2 : 0 \le a < 1, 0 \le b < 1\}$.

(b) Prove that $\wp(z) - c$ has a single double zero in Π' exactly when c is $e_1 = \wp(\omega_1/2)$, $e_2 = \wp(\omega_2/2)$ or $e_3 = \wp((\omega_1 + \omega_2)/2)$.

(c) Conclude that $(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$.

(3) Prove that for $k \ge 4$, $E_k(z) = \dfrac{1}{2} \displaystyle\sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} \frac{1}{(mz+n)^k}$.

(4) (a) Prove that

$$E_2(\gamma z) = (cz + d)^2 E_2(z) + \frac{12c}{2\pi i}(cz + d).$$

You may use the fact that

$$E_2(-1/z) = z^2 E_2(z) + \frac{12z}{2\pi i}$$

without proof.

(b) Let f be a modular form of weight k for Γ, and define

$$g(z) := \frac{1}{2\pi i} f'(z) - \frac{k}{12} E_2(z) f(z).$$

Prove that $g(z)$ is a weight $k+2$ modular form for Γ, and that it is a cusp form if and only if f is a cusp form.

(5) Does there exist $f(z) \in S_{50}(\Gamma)$ whose Fourier expansion starts as follows?

(a) $q + 1391q^2 + 611387q^3 - 65464592q^4 - 177273987830q^5 + 50038976872422q^6$.

(b) $2008q + 2795131q^2 + 1228965068q^3 - 131587216304q^4 - 356296628154400q^5$.