NEUKIRCH, EXERCISE I.4.9

MATH 105

Let $\mathcal{O}$ be a domain in which every nonzero ideal can be factored into a (unique) product of prime ideals, and let K be its field of fractions. We will show that $\mathcal{O}$ is a Dedekind domain.

(a) A *fractional ideal* $\mathfrak{a}$ of $\mathcal{O}$ is a nonzero $\mathcal{O}$-submodule of K such that there exists nonzero $d \in \mathcal{O}$ such that $d\mathfrak{a} \in \mathcal{O}$. A fractional ideal $\mathfrak{a}$ is *invertible* if there exists a fractional ideal $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} = \mathcal{O}$. Show that if a fractional ideal is invertible, then the inverse is unique and it is equal to

$$\mathfrak{a}^{-1} = \{x \in K : x\mathfrak{a} \subseteq \mathcal{O}\}.$$

(b) Show that any prime factor of a principal ideal is invertible, and that any factorization of an ideal into invertible ideals is unique.

(c) Show that every nonzero prime ideal of $\mathcal{O}$ is invertible, and conclude that every nonzero fractional ideal of $\mathcal{O}$ is invertible.

(c1) Let $p \in \mathfrak{p}$ be nonzero, and conclude that $\mathfrak{q} \subseteq (p) \subseteq \mathfrak{p}$ with $\mathfrak{q}$ invertible.

(c2) Let $a \in \mathfrak{p} \setminus \mathfrak{q}$, and consider the factorization of the ideals $\mathfrak{q} + a\mathcal{O}$ and $\mathfrak{q} + a^2\mathcal{O}$. Show that every such prime factor contains $\mathfrak{q}$, so we can consider these factorizations in the quotient ring $\mathcal{O}/\mathfrak{q}$. Conclude by unique factorization that $\mathfrak{q} + a^2\mathcal{O} = (\mathfrak{q} + a\mathcal{O})^2$.

(c3) From

$$\mathfrak{q} \subseteq \mathfrak{q} + a^2\mathcal{O} = (\mathfrak{q} + a\mathcal{O})^2 \subseteq \mathfrak{q}^2 + a\mathcal{O}$$

show that

$$\mathfrak{q} \subseteq \mathfrak{q}^2 + a\mathfrak{q}$$

and then that equality holds.

(c4) From the invertibility of $\mathfrak{q}$, derive a contradiction; conclude that $\mathfrak{q} = \mathfrak{p}$ and thus $\mathfrak{p}$ is invertible.

(d) Show that $\mathcal{O}$ is integrally closed. *[Hint: if $\alpha \in K$ is integral, then the ring $\mathcal{O}[\alpha]$ is a fractional ideal of $\mathcal{O}$ and $\alpha\mathcal{O}[\alpha] \subseteq \mathcal{O}[\alpha]$.]*

(e) Let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals of $\mathcal{O}$. Show that $\mathfrak{a} \supseteq \mathfrak{b}$ if and only if there exists a fractional ideal $\mathfrak{q}$ such that $\mathfrak{b} = \mathfrak{q}\mathfrak{a}$. *[Hint: Reduce to the case where $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}$. Argue by induction on the number of prime factors dividing $\mathfrak{a}$.]*

(f) Conclude that $\mathcal{O}$ is Noetherian and every prime ideal is maximal.