Math 103: Measure Theory and Complex Analysis
Fall 2018

11/2/18

Lecture 22

Outline: Using **Cauchy's Formula for Convex Sets** applied to disks we can now prove that $f \in \mathcal{H}(\Omega) \Leftrightarrow f$ analytic in Ω.

Recall: If $f : \Omega \to \mathbb{C}$ holomorphic and $\gamma \subset \Omega$ is a closed path then for $z \in \Omega \backslash \gamma^\star$ then

$$\mathrm{Ind}_\gamma(z) \cdot f(z) = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{w - z}\, dw.$$

Picture

Theorem 5 (analytic = holomorphic) Let Ω be a domain and $f : \Omega \to \mathbb{C}$ a function. Then

$$f \in \mathcal{H}(\Omega) \Leftrightarrow f \text{ is analytic in } \Omega$$

In particular, $f \in \mathcal{H}(\Omega) \Rightarrow f' \in \mathcal{H}(\Omega)$. Furthermore if Ω is convex then $\displaystyle\int f$ in $\mathcal{H}(\Omega)$ by **Cauchy's theorem for Convex Sets**.

Note: The condition that Ω is convex can be extended to Ω simply connected, but not completely dropped. The example is the function

$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, z \mapsto \frac{1}{z}.$$

It's antiderivative is $\log(z)$, which is defined on $\mathbb{C} \setminus \{0\}$ but by our definition not continuous on the real axis. We can understand the function $\frac{1}{z}$ by lookingat the images of circles and using polar coordinates i.e. $z = r \cdot e^{i\varphi}$. Then $\frac{1}{z} =$.

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proof: We already have that analytic $\Rightarrow$ holomorphic. Now assume that f is holomorphic in Ω and let $a \in \Omega, R > 0$ such that $D_R(a) \subseteq \Omega$. Choose r such that $0 < r < R$ and let

$$\gamma(t) = a + re^{it} \text{ for } t \in [0, 2\pi)$$

Picture

Then if $z \in D_r(a)$, $\mathrm{Ind}_\gamma(z) = 1$ and by Cauchy's Formula,

$$f(z) = \qquad (1.)$$

The RHS is of the form $\int_X \frac{1}{\varphi(t)-z} d\nu(t)$ with

$$\begin{aligned} X &= \\ \varphi(t) &= \\ d\nu(t) &= \qquad\qquad dt \end{aligned}$$

so f is analytic by **Ch.1, Theorem 12**. $\square$

Note: The formula for the derivatives of f obtained in **Ch. 1, Theorem 12** is equal to deriving $f(z)$ in (1) under the integral. Hence we obtain

$$f^{(k)}(z) = \qquad .$$

Remark 6 Note that by our proof the power series expansion for $f \in \mathcal{H}(\Omega)$ about $a \in \Omega$ converges in the largest disk $D_R(a)$ such that $D_R(a) \subseteq \Omega$.

Example Consider the power series expansion for f around 0 where

$$\begin{aligned} f(z) &= \frac{1}{1+e^z} \\ &= \frac{1}{2} - \frac{1}{4}z + \frac{1}{48}z^3 - \frac{1}{480}z^5 + o(z^7) \end{aligned}$$

Then f is not defined for . Hence f is holomorphic on .
Hence the radius of convergence is .

Picture Sketch the domain of f and $|f|$ on the xy plane exchanging the y and x axis.

Corollary 7 If Ω is a convex region, $f \in \mathcal{H}(\Omega)$ and γ is a closed path in Ω, then for any $z \in \Omega \setminus \gamma^\star$

$$\text{Ind}_\gamma(z) \cdot f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{k+1}} dw$$

proof: If

$$g(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw$$

then we know that

$$g^{(k)}(z) =$$

Then we can set $g(z) = \text{Ind}_\gamma(z) \cdot f(z)$ and Ind_γ is constant on the components of $\Omega \setminus \gamma^\star$. $\square$

Theorem 8 (Morera's Theorem) Suppose that f is continuous on a domain Ω and that

$$\int_{\partial\Delta} f(z) dz = 0$$

for all triangles $\Delta \subseteq \Omega$. Then $f \in \mathcal{H}(\Omega)$. This is a converse of sorts for Cauchy's Theorem.

proof: It suffices to prove that f is holomorphic in every open disk in Ω. Let V be a convex subregion of Ω and $a \in V$.

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Picture

For $z \in V$ let

$$F(z) = \int_{[a,z]} f(w)dw$$

Then $F' = f$:

But $F \in \mathcal{H}(V)$, so $F' = f \in \mathcal{H}(V)$. $\square$

Definition 9 If $a \in \mathbb{C}$ and $r > 0$ then

$$D_r'(a) := \{z : 0 < |z - a| < r\} = D_r(a) \setminus \{a\}$$

is called the **punctured disk** of radius r centered at a.
Picture

Definition 10 If $E \subseteq \mathbb{C}$, then p is a **limit point** of E if

$$\forall r > 0, D_r'(p) \cap E \neq \emptyset$$

We denote by E' the set of limit points of E.

Observe E' is closed (its complement is clearly open).

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Theorem 11 Let Ω be a region and $f \in \mathcal{H}(\Omega)$. Let $Z(f) = \{a \in \Omega : f(a) = 0\}$. Then either

$$Z(f) = \Omega \quad \text{or} \quad Z(f)' \cap \Omega = \emptyset$$

In the latter case, given $a \in Z(f)$, $\exists m$ such that $\exists g \in \mathcal{H}(\Omega)$ with $g(a) \neq 0$ and

$$f(z) = (z - a)^m g(z)$$

for all $z \in \Omega$. We call m the **order** of the zero a.

Note This is analogous to polynomial behavior.

Note If $f'(a) = 0$, then the angle between two vectors is multiplied in the image.