Math 103: Measure Theory and Complex Analysis
Fall 2018

10/31/18

Lecture 21

Chapter 3 - The local Cauchy Theorem

Idea: γ closed in Ω, $f \in \mathcal{H}(\Omega)$ + topological condition on $\Omega \Longrightarrow \int_\gamma f\, dt = 0$.

Theorem 1 (Cauchy's Theorem for triangles) Let Ω be a domain and $\triangle = \triangle(a, b, c) \subseteq \Omega$. If $f : \Omega \to \mathbb{C}$ is continuous and $f \in \mathcal{H}(\Omega \setminus \{p\})$, then

$$\int_{\partial\triangle} f(z)\, dz = 0.$$

Picture

proof Assume first that $\boxed{p \notin \triangle}$. If a, b, c lie on a straight line, then this is true. Otherwise, let a', b', c' be the midpoints of $[b, c]$, $[a, c]$ and $[a, b]$, respectively and set:

$$\begin{aligned}
\triangle^1 &= \triangle(ac'b') \\
\triangle^2 &= \triangle(ba'c') \\
\triangle^3 &= \triangle(cb'a') \\
\triangle^4 &= \triangle(a'b'c')
\end{aligned}$$

Let

$$J = \int_{\partial\triangle} f(z)\, dz = \sum_{k=1}^{4} \int_{\partial\triangle^k} f(z)\, dz \qquad \text{and} \qquad L = \ell(\partial\triangle).$$

We want to show that $\boxed{|J| = 0}$. We know that for some $k \in \{1, 2, 3, 4\}$, we have

$|J| \leq$, $\ell(\partial\triangle^k) =$ and $\operatorname{diam}(\triangle) \leq$

Set $\triangle^k = \triangle_1$, and iterate: $\triangle \supset \triangle_1 \supset \triangle_2 \cdots$.

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Note: (1) $l(\partial\triangle_n) =$

(2) $|J| \leq$

(3) $\text{diam}(\triangle_n) =$

Now $\triangle$ is compact and each $\triangle_n$ is compact. We can choose a point $z_n \in \triangle_n$. Then $(z_n)_n$ is a Cauchy sequence. Let $z_0 = \lim_{n\to\infty} z_n$. Then

$$\bigcap_{n\geq 1} \triangle_n = \{z_0\}, \quad z_0 \in \triangle.$$

Picture

Since we are assuming $p \notin \triangle$, $f'(z_0)$ exists: $\forall\varepsilon > 0$, $\exists\delta > 0$ such that

Choose n large enough, so $z \in \triangle_n \Rightarrow |z - z_0| < \delta$.

By **Corollary 12** of the Fundamental Theorem we have,

$$\int_{\partial\triangle_n} f(z)\,dz = \int_{\partial\triangle^n} \big(f(z) - f(z_0) - f'(z_0)(z - z_0)\big)\,dz.$$

So using (1) - (3) we get:

$$\left|\int_{\partial\triangle^n} f(z)\,dz\right| \leq$$

Picture

Now assume $\boxed{p \text{ is a vertex}}$. We may assume $p = a$. Since f is bounded, $\exists\, x \in [a, b]$ and $\exists\, y \in [c, a]$ such that

$$\left| \int_{\partial\triangle(a,x,y)} f(z)\, dz \right| < \varepsilon.$$

But

$$J =$$

and $\partial\triangle(a, x, y)$ can be made arbitrarily small if $x, y \to a$.

Finally if $\boxed{p \in \triangle, \text{ but not a vertex}}$, we triangulate $\triangle$ with the new vertex p (see above). This settles the proof in the last case. $\square$

Theorem 2 (Cauchy's Theorem for Convex Sets) Suppose Ω is a convex region, that f is continuous on Ω and $f \in \mathcal{H}(\Omega \setminus \{p\})$ for some $p \in \Omega$. Then

$$\int_\gamma f(z)\, dz = 0 \quad \text{for any closed path} \quad \gamma \subset \Omega.$$

In fact, f has an antiderivative in Ω.

Picture

Picture

proof: Fix $a \in \Omega$ and defined, for $z \in \Omega$,

$$F(z) = \int_{[a,z]} f(w)dw$$

Notice that since Ω is convex, if $z, z_0 \in \Omega$, then $\triangle(a, z, z_0) \subseteq \Omega$.

Therefore, by **Cauchy's Theorem for triangles**, we get

$$F(z) = \int_{[a,z_0]} f(w)\,dw + \int_{[z_0,z]} f(w)\,dw.$$

Hence if $z \neq z_0$ in Ω, then using that $f(w) = f(w) - f(z_0) + f(z_0)$ we have

$$\frac{F(z) - F(z_0)}{z - z_0} =$$

Since f is continuous at z_0, we have for fixed $\epsilon > 0$ there is $\delta > 0$, such that

Therefore

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) =$$

Hence $F \in \mathcal{H}(\Omega)$ and $F' = f$.
The conclusion follows from the **Fundamental Theorem for Line Integrals**. $\square$

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Theorem 3 (Cauchy's Formula in a Convex Set) Suppose Ω is a convex region and $f \in \mathcal{H}(\Omega)$. Let γ be a closed path in Ω. If $z \in \Omega \setminus \gamma^*$, then

$$\text{Ind}_\gamma(z) \cdot f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw$$

Picture

Remark 4 This is most useful if γ^* is a circle or a Jordan curve and z is "inside" of γ with $\text{Ind}_\gamma(z) = 1$. If $f \equiv 1$, this is true by definition of the index.

proof: Fix $z \in \Omega \setminus \gamma^*$ and let

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z \\ f'(z) & \text{if } w = z \end{cases}$$

It is continuous on Ω and $g \in \mathcal{H}(\Omega \setminus \{z\})$. By Cauchy's theorem, $\int_\gamma g = 0$, that is

$\square$