Math 103: Measure Theory and Complex Analysis
Fall 2018

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Lecture 19

Corollary 10 Suppose f is analytic in Ω. Then f has derivatives of all orders in Ω, each of which is analytic in Ω.

Corollary 11 If f is analytic in Ω and $f(z) = \sum_{n\geq 0} a_n(z-a)^n$ for all $z \in D_r(a)$ then $a_k = \frac{f^{(k)}(a)}{k!}$. In particular, the power series expansion at a is unique.

proof $f^{(k)}(z) =$

We describe a process that produces analytic functions.

Theorem 12 (Analytic functions from integrals) Let ν be a complex measure on a measurable set $(X, \mathcal{M})$, with $|\nu|(X) < \infty$, and let $\varphi : X \to \mathbb{C}$ be a measurable function and $\Omega \subseteq \mathbb{C}$ a domain such that $\varphi(X) \cap \Omega = \emptyset$. Then the function

$$f(z) = \int_X \frac{1}{\varphi(x) - z} d\nu(x)$$

is analytic in Ω. Moreover, $f^{(k)}(z) = k! \int_X \frac{1}{(\varphi - z)^{k+1}} d\nu$ for $k \in \mathbb{N}$.

Picture

proof Let $a \in \Omega$ and $r > 0$ such that $D_r(a) \subseteq \Omega$. Note that if $z \in D_r(a)$ and $x \in X$ then

$$\left| \frac{z-a}{\varphi(x) - a} \right| \leq$$

Looking at the geometric series $\sum_{m\geq 0} q^m$ with $q = \frac{z-a}{\varphi(x)-a}$, we see

$$\sum_{m\geq 0} \frac{(z-a)^m}{(\varphi(x) - a)^m} =$$

This means that the series converges for $|z-a| \leq r$ and

$$\frac{1}{\varphi(x)-z} =$$

and the convergence is uniform in X for each $z \in D_r(a)$.

Since $|\nu|(X) < \infty$,

$$f(z) = \int_X \frac{1}{\varphi(x)-z} d\nu(x) =$$

and the right hand side convergence for all $z \in D_r(a)$. Therefore, f is analytic in Ω and

$$f^{(k)}(a) = \qquad \square$$

Corollary 13 In the previous theorem, the power series for f about $a \in \Omega$ converges in any disc $D_r(a)$ contained in Ω.

Chapter 2 - Curves and integrals over curves

Definition 1 If X is a topological space, a **curve** in X is a continuous map $\gamma : [a,b] \to X$. The image $\gamma([a,b])$ is denoted by $\gamma^\star$. If $\gamma(a) = \gamma(b)$, we say that γ is closed.

Note $\gamma_1^\star = \gamma_2^\star \not\Longrightarrow \gamma_1 = \gamma_2$. Also, there are surjective maps $[0,1] \to [0,1]^2$.

Definition 2 A closed curve γ in X is called **simple** if

$$a \leq t < s < b \implies \gamma(t) \neq \gamma(s).$$

Picture

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Theorem 3 (Jordan Curve Theorem) The complement of a simple closed curve γ in $\mathbb{C}$ consists of two open connected components, one of which is bounded and both of which have $\gamma^\star$ as their common boundary.

Picture

Definition 4 A **path** in $\mathbb{C}$ is a piecewise continuously differentiable curve $\gamma : [a, b] \to \mathbb{C}$. Thus, there exists a subdivision $\mathcal{D} = \{a = t_0 < t_1 < \ldots < t_n < b\}$ of $[a, b]$ such that γ' is continuous on $[t_{i-1}, t_i]$ for $i \in \{1, \ldots, n\}$.

One-sided derivatives exist at each t_i.

Definition 5 If $\gamma : [a, b] \to \mathbb{C}$ is a path and $f : \gamma^\star \to \mathbb{C}$ is continuous, we define

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

The **length** of γ is

$$\ell(\gamma) := \int_a^b \left|\gamma'(t)\right| dt.$$

Definition 6 Two paths γ_1 and γ_2 with $\gamma_1^\star = \gamma_2^\star := \gamma^\star$ are **equivalent** if for all $f \in C(\gamma^\star)$, we have

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

Example (Reparametrization) Let $\gamma : [a, b] \to \mathbb{C}$ be a path and $\varphi : [c, d] \to [a, b]$ a bijective continuously differentiable map. Then $\gamma \circ \varphi : [c, d] \to \mathbb{C}$ is equivalent to γ.

Remark 7

a) Every path can be reparametrized, such that $[a,b] = [0,1]$.

b) If γ_1, γ_2 are paths such that the terminal point of γ_1 is the initial point of γ_2, then there is a path $\gamma_1 + \gamma_2$, called the **join** of γ_1 and γ_2, such that

$$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz \quad \text{for all} \quad f \in C(\gamma_1^* \cup \gamma_1^*).$$

Picture

c) If $\gamma : [a,b] \to \mathbb{C}$ is a path, then there is an **inverse path** $-\gamma : [a,b] \to \mathbb{C}$ given by

$$-\gamma(t) = \gamma(a+b-t), \quad \text{such that} \quad \int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz \quad \text{for all} \quad f \in C(\gamma^*).$$

d) If $u, w \in \mathbb{C}$, let $[u,w]$ be the path $t \to u + t(w-u)$, for $t \in [0,1]$ which parametrizes the **line segment** between u and w. Then

$$\int_{[u,w]} f(z)\,dz = (w-u) \cdot \int_0^1 f(u+t(w-u))\,dt \quad \text{for all} \quad f \in C([u,w]).$$

e) If $a, b, c \in \mathbb{C}$, then $\triangle(a,b,c) = \{\lambda_1 a + \lambda_2 b + \lambda_3 c, \text{ where } \lambda_i \geq 0, \lambda_1 + \lambda_2 + \lambda_3 = 1\}$. Note that

$$\int_{\partial\triangle(a,bc)} f(z)\,dz = \int_{[a,b]} f(z)\,dz + \int_{[b,c]} f(z)\,dz + \int_{[c,a]} f(z)\,dz.$$

Here the left-hand side is invariant under cyclic permutations of the (a,b,c) and only changes sign if (a,b,c) is replaced by (a,c,b).

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Reminder $e^{x+iy} \overset{Def.}{=} e^x \cdot (\cos(y) + i\sin(y))$. Then

$$f(z) = e^z \in \mathcal{H}(\mathbb{C}) \text{ and } f' = f.$$

Picture Plot e^z using the grid map i.e. look at $\{f(x+iy), x = const.\}$ and $\{f(x+iy), y = const.\}$

Furthermore $e^w = 1 \leftrightarrow w = 2\pi \cdot i \cdot k$ with $k \in \mathbb{Z}$.

Remark 8 For any continuous $f : \Omega \to \mathbb{C}$ we have

$$\left|\int_\gamma f(z)\,dz\right| \leq \int_\gamma |f(z)|\,dz \leq M\ell(\gamma) \quad \text{where} \quad M = \max\{|f(z)|, z \in \gamma^*\}.$$

Theorem 9 Let γ be a closed path and $\Omega = \mathbb{C} \setminus \gamma^*$. If $z \in \Omega$, define

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2i\pi} \cdot \int_\gamma \frac{1}{w - z}\,dw$$

Then $\mathrm{Ind}_\gamma : \Omega \to \mathbb{Z}$ is constant on connected components of Ω and 0 on the unbounded component.
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