Math 103: Measure Theory and Complex Analysis
Fall 2018

10/24/18

Lecture 18

Part II - COMPLEX ANALYSIS

Outline In real analysis, everything that could go wrong does. In complex analysis, everything that you dream of is true.

Chapter 1 - Holomorphic functions

Definition 1 A subset $A \subset \mathbb{C}$ is called **connected** if it is **not** the union of two disjoint non-empty open sets.
Picture

Definition 2 An open subset $\Omega \subseteq \mathbb{C}$ is called a **domain**. A nonempty connected domain A is called a **region**.

Definition 3 If $E \subseteq \mathbb{C}$ and $x \in E$ then the **connected component** of x in E is

$$C(x) = \bigcup \{A \subseteq \mathbb{C} \mid x \in A \subseteq E, A \text{ is connected}\}$$

Note that $C(x)$ is connected and $C(x) \cap C(y) \neq \emptyset \Leftrightarrow C(x) = C(y)$. Since

$$D_r(a) = \{z \in \mathbb{C} \mid |z - a| < r\}$$

is connected, it follows that a connected component of any domain is the union of disks and therefore open. In other words, **every domain is a union of regions**.

Definition 4 Let $\Omega \subseteq \mathbb{C}$ be a domain and $z \in \Omega$. We say that $f : \Omega \to \mathbb{C}$ is **differentiable** at z if

$$f'(z) = L = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \qquad \text{exists.}$$

This means for all $\epsilon > 0$ there is $\delta > 0$, such that

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Definiton 5 We say that f is **holomorphic** on Ω if $f'(z)$ exists for all $z \in \Omega$. Denote by $\mathcal{H}(\Omega)$ the collection of all holomorphic functions on Ω.

Interpreting $z = x + iy = (x, y)$ as a point in $\mathbb{R}^2 \simeq \mathbb{C}$. We can see any function $f : \mathbb{C} \to \mathbb{C}$ as a deformation or transformation of the plane: This can be seen by splitting f into the real and imaginary part

$$\begin{aligned} f(z) = f(x+iy) = f(x,y) &= \operatorname{Re}(f)(x,y) + i\operatorname{Im}(f)(x,y) \\ &= (\operatorname{Re}(f)(x,y), \operatorname{Im}(f)(x,y)) = (u(x,y), v(x,y)). \end{aligned}$$

Picture

For fixed $z = x + iy$ and for $t \in \mathbb{R}$ we take once $h = t$ and $h = it$ in **Definition 5**. Writing $f = \operatorname{Re}(f) + i\operatorname{Im}(f) = u + iv$ we get for a holomorphic function:

$$\begin{aligned} f'(z) = \lim_{t\to 0} \frac{f(z+t) - f(z)}{t} &= \\ &= \\ f'(z) = \lim_{t\to 0} \frac{f(z+it) - f(z)}{it} &= \\ &= \qquad . \end{aligned}$$

Hence

$$\boxed{\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \quad \text{and} \quad \frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial y}.}$$

These are the **Cauchy-Riemann Equations**.

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The converse is also true:

Theorem 6 (Cauchy-Riemann Equations (CRE)) Let $f : \Omega \to \mathbb{C}$ is a holomorphic function, where $f = (u, v) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ if and only if $f\, \mathbb{C} \simeq \mathbb{R}^2 \to \mathbb{R}^2$ is real differentiable and

$$Df = D(u, v) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{and} \quad \det(Df) = a^2 + b^2.$$

proof The proof is an exercise.

Conformal maps

Conformal maps preserve the angles of intersecting curves. We will show that holomorphic maps are conformal maps.

Picture

For two curves $\gamma, \delta : [-1, 1] \to \mathbb{C} \simeq \mathbb{R}^2$ with $\gamma(0) = \delta(0) = p$ and $v = \gamma'(0), w = \delta'(0)$ we get two curves $f \circ \gamma$ and $f \circ \delta$ in the image. Then

$$\begin{aligned} (f \circ \gamma)'(0) &= Df \mid_{\gamma(0)} \cdot \gamma'(0) = Df(p) \cdot v \\ (f \circ \delta)'(0) &= Df \mid_{\delta(0)} \cdot \delta'(0) = Df(p) \cdot w. \end{aligned}$$

For the dot product we obtain

$$\begin{aligned} (f \circ \gamma)'(0) \bullet (f \circ \delta)'(0) &= (Df(p) \cdot v)^T \cdot Df(p) \cdot w \\ &= v^T Df(p)^T \cdot Df(p) \cdot w = \det(Df(p))(v \bullet w). \end{aligned}$$

Looking at the angles between the curves we get

$$\begin{aligned} \cos(\angle(v, w)) &= \\ \cos(\angle(Df(p)v, Df(p)w)) &= \end{aligned}$$

Hence f preserves the angles of intersection of curves at any point p, where $Df(p) \neq 0$.

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Conversely if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a conformal map, such that $Df(p) \neq 0$ then f it either preserves the orientation of vectors in which case it is holomorphic or it exchanges the orientation of vectors in which case it is anti-holomorphic.

Picture

We have proven:
Theorem 7 If $f : \Omega \to \mathbb{C}$ holomorphic, then f is conformal on $\Omega \backslash \{p \in \Omega : Df(p) \neq 0\}$.

Reminder on power series

If $z \in \mathbb{C}$ and $\{a_n\}_{n\geq 0}$ is a sequence in $\mathbb{C}$,

$$\sum_{n\geq 0} a_n (z - z_0)^n$$

has a radius of convergence $R \in [0, \infty]$ such that the series converges uniformly and absolutely on $D_r(z_0)$ provided that $0 \leq r < R$ and diverges for all $z \notin \overline{D_R(z_0)}$. In fact,

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \limsup_{n\geq 0} |a_n|^{1/n}.$$

Definition 8 We say that $f : \Omega \to \mathbb{C}$ is **analytic** in a domain Ω if for all $a \in \Omega$ there is an $r > 0$ such that $D_r \subseteq \Omega$ and there exists $\{a_n\}_{n\geq 0}$ such that for all $z \in D_r(a)$ we have

$$f(z) = \sum_{n\geq 0} a_n (z - a)^n.$$

Out first goal will be to show that f is holomorphic on Ω if and only if f is analytic on Ω.

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Theorem 9 Suppose that f is analytic on Ω. Then $f \in \mathcal{H}(\Omega)$ and f' is analytic in Ω. In fact, if $f(z) = \sum_{n\geq 0} a_n(z-a)^n$ for $z \in D_r(a)$, then

$$f'(z) = \sum_{n\geq 1} na_n(z-a)^{n-1} \quad \text{for} \quad z \in D_r(a).$$

Picture

proof We first note that both power series have the same radius of convergence:

Let

$$g(z) = \sum_{n\geq 1} na_n(z-a)^{n-1} \quad \text{for} \quad z \in D_r(a).$$

Replacing $z-a$ with z if necessary, we may assume $a = 0$. Fix $w \in D_r(0)$ and $\rho > 0$ such that $\boxed{0 \leq |w| < \rho < r}$. If $z \neq w$ then

$$\frac{f(z)-f(w)}{z-w} - g(w) =$$

$$= \sum_{n\geq 1} a_n A_n$$

with $A_1 =$ and $A_n =$ if $n > 1$. One can check that if $n > 1$ then

$$A_n = (z-w)\sum_{k=1}^{n-1} kw^{k-1}z^{n-k-1}.$$

Thus if $|z| < \rho$ and $|w| < \rho$, we see that

$$|A_n| \leq$$

Thus

$$\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| \leq \qquad .$$

Since $\limsup_n |a_n|^{1/n} = \frac{1}{R}$ and $\rho < r \leq R$ we see that the series on the right-hand side converges by the root test. Therefore, $\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| \leq |z-w|\, M$ as long as $|z| < \rho$. Thus $f'(w)$ exists and equals $g(w)$. $\square$
