Math 103: Measure Theory and Complex Analysis
Fall 2018

10/22/18

Lecture 17

Chapter 2.8. - The Radon-Nikodym Theorem

Outline Let $(X, \mathcal{M}, \mu)$ be a measure space. We show that under certain simple conditions we have for another measure $\nu \ll \mu$ and $E \in \mathcal{M}$

$$\nu(E) = \int_E f \, d\mu, \quad \text{where } f : X \to [0, \infty) \text{ measurable.}$$

In a sense this means that measuring and integrating are the same thing.

Picture

We recall

Ch. 1.6, Theorem 10 Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to [0, \infty)$ be a measurable function. Then there is a measure μ_f on X given by

$$\begin{aligned} \mu_f : \mathcal{M} &\to [0, \infty] \\ E \mapsto \mu_f(E) &= \int_E f d\mu \end{aligned}$$

Moreover, if g is measurable on X then

$$\int_X g \, d\mu_f = \int_X gf \, d\mu.$$

Definition 1 Let μ and ν be measures on a measurable set $(X, \mathcal{M})$. We say that ν is **absolutely continuous** with respect to μ and we write $\nu \ll \mu$ if $\boxed{\mu(E) = 0 \Rightarrow \nu(E) = 0}$.

Note 2 In **Chapter 1.6.** we have also shown that $\mu_f \ll \mu$.

Example If f is a probability density then $\mu_f \ll \mu$.

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Theorem 3 (Radon-Nikodym) If μ and ν are finite measures on $(X, \mathcal{M})$ such that $\nu \ll \mu$ then there exists a measurable function

$$f : X \to [0, \infty) \text{ such that } \nu = \mu_f.$$

If g is any function such that $\nu = \mu_g$, then $f = g$ almost everywhere (with respect to μ).

proof Idea: The idea is to construct explicitly a function f that satisfies the conditions of the theorem. We will make use of the **Hahn decomposition**. We first consider the case where both measures are finite.

1.) $\mu(X) < \infty$ **and** $\nu(X) < \infty$.

a) Partitioning X

We first divide up X into suitable sets, where an approximation of f can be defined by simple functions. Fix $c > 0$. Then $\nu - c\mu$ is a signed measure. Let $\{P(c), N(c)\}$ be a Hahn decomposition for $\boxed{\nu - c\mu}$. We have:

$$c_2 \geq c_1 \Rightarrow \qquad\qquad \text{for all } E \in \mathcal{M}.$$

Picture

Now consider $\bigcup_{k \geq 1} N(kc)$ and make it disjoint. We set:

$$A_1 = N(c)$$
$$A_k = N(kc) \setminus \bigcup_{j<k} N(jc) = \qquad\qquad .$$

We see

$$\bigcup_{k \geq 1} N(kc) = \biguplus_{k \geq 1} A_k.$$

If $E \subset A_k$ and $E \in \mathcal{M}$ then

$$\begin{aligned} E &\subseteq N(kc) \text{ so} \\ E &\subseteq P((k-1)c) \text{ so} \end{aligned} \quad , \quad \text{hence}$$

$$\boxed{(k-1)c\mu(E) \leq \nu(E) \leq kc\mu(E).} \tag{1}$$

This means that heuristically $(k-1)c\mu \leq \nu \leq kc\mu$ on A_k. Let

$$B = X \setminus \biguplus_{k \geq 1} A_k =$$

Since for any $k \in \mathbb{N}$ we have $B \subset P(kc)$ and therefore $0 \leq \nu(B) - kc\mu(B)$. Hence

As k may be chosen to be arbitrarily large, this implies $\mu(B) = 0$ and therefore $\nu(B) = 0$ since $\nu \ll \mu$.

b) Construction of f

We will use (1) to construct a function f that satisfies the conditions of the theorem. Let

$$g_c(x) = \begin{cases} (k-1)c & \text{if } x \in A_k \\ 0 & \text{if } x \in B. \end{cases}$$

We see that $g_c = \sum_{k \geq 1} (k-1)c\mathbb{1}_{A_k}$. Then for all $E \in \mathcal{M}$, we have by (1)

$$\int_E g_c d\mu \leq \qquad . \tag{2}$$

We now make a "refinement" using the parameter c. To this end let $\boxed{f_n = g_{2^{-n}}}$, and assume $m \leq n$ in $\mathbb{N}$. We want to show that $(f_n)_n$ converges. To this end we note that by (2)

$$\begin{aligned} \int_E f_n d\mu \leq \nu(E) \leq & \qquad \text{and} \\ \int_E f_m d\mu \leq \nu(E) \leq & \end{aligned} \tag{3}$$

so, as $2^{-n} \leq 2^{-m}$ we have

$$\left| \int_E (f_n - f_m) d\mu \right| \leq \qquad .$$

Apply this with $E = E_+ := \{x \in X \mid f_n(x) - f_m(x) \geq 0\}$ and $E = E_- := \{x \in X \mid f_n(x) - f_m(x) < 0\}$ to conclude

$$\int_X |f_n - f_m| \, d\mu \leq \qquad\qquad\qquad\qquad .$$

In other words, $(f_n)_{n\geq 1}$ is a Cauchy sequence in $L^1(X, \mathcal{M}, \mu)$. Therefore, by **Ch. 2.6. Prop. 6,7** we can extract a subsequence $(f_{n_k})_{k\geq 1}$ such that $f_{n_k} \underset{k\to\infty}{\to} f$ almost everywhere. Thus we can assume $f(x) \geq 0$ for each $x \in X$.

$$\left| \int_E f_n d\mu - \int_E f d\mu \right| \leq$$

As the latter goes to zero for n to infinity by the $\triangle \neq$ we have that

$$\int_E f_n d\mu \underset{n\to\infty}{\to} \int_E f d\mu.$$

Returning to (3):

$$\nu(E) = \lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu.$$

Can you prove uniqueness?

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2.) ν and μ are σ finite

Now we extend the result to the σ-finite case: Assume that

$$X = \bigcup_{n\geq 1} X_n \quad \text{with} \quad X_n \subset X_{n+1} \quad \text{and} \quad \nu(X_n) < \infty, \mu(X_n) < \infty \quad \text{for all} \quad n \in \mathbb{N}\,.$$

We know that we can find $h_n : X \to [0,\infty)$ such that

1. $h_n(x)|_{X_n^C} \equiv 0$,
2. For all $E \in \mathcal{M}$, $E \subset X_n$ implies $\nu(E) = \int\limits_{X_n} h_n d\mu$.

Now, if $\boxed{n \leq m}$ and $E \subseteq X_n$, then $\int\limits_E h_n d\mu = \int\limits_E h_m d\mu$.

Picture

Thus $h_n|_{X_n} = h_m|_{X_n}$ almost everywhere. Let $f_n(x) = \max\{h_1(x), \ldots, h_n(x)\} = h_n(x)$ almost everywhere with respect to μ. Then $f_n \nearrow f : X \to [0,\infty]$. If $E \in \mathcal{M}$ then

$$\begin{aligned}\nu(E) &= \lim_{n\to\infty} \nu(E \cap X_n)\\ &= \\ &= \\ &= \qquad\qquad\qquad\qquad .\end{aligned}$$

Now let $A = \{x \mid f(x) = +\infty\}$. We see $\mu(A \cap X_n) = 0$ (otherwise $\nu(A \cap X_n) = \infty$). Thus $\mu(A) = \lim\limits_{n\to\infty} \mu(A \cap X_n) = 0$ and we can assume $f : X \to [0,\infty)$. This completes our proof. $\square$