

Math 73/103: Measure Theory and Complex Analysis
Fall 2019 - Homework 4

The following problems reprise some of the fundamental results about functions of a complex variable covered in elementary courses but not covered in Chapter 10 of [Rudin]. Most of this material — with perhaps the exception of Question 4 — are part of the early chapters in basic texts such as Conway, Brown & Churchill or Saff & Snider.

Recall: Let Ω be a domain in \mathbb{C} and assume that $f : \Omega \rightarrow \mathbb{C}$ is a function. Of course, we can view Ω as an open subset of \mathbb{R}^2 and define $u, v : \Omega \rightarrow \mathbb{R}$ by

$$u(x, y) := \operatorname{Re}(f(x + iy)) \quad \text{and} \quad v(x, y) = \operatorname{Im}(f(x + iy))$$

We say that the *Cauchy-Riemann Equations hold at* $z_0 = x_0 + iy_0$ if the partial derivatives of u and v exist at (x_0, y_0) and

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0). \quad (\text{CRE})$$

We often abuse notation slightly, and say that (CRE) amounts to $f_y(z_0) = if_x(z_0)$. (Just to be specific, $f_x(x_0 + iy_0) := u_x(x_0, y_0) + iv_x(x_0, y_0)$.)

1. Suppose that Ω is a *region* in \mathbb{C} , and that $f \in \mathcal{H}(\Omega)$. Show that if $f'(z) = 0$ for all $z \in \Omega$, then f is constant.

Hint: You can prove for yourself or use without proof that if $u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is such that $u_x(x, y) = u_y(x, y)$ for all $(x, y) \in \Omega$ then u is constant — provided Ω is a region.

2. Suppose that Ω is a region and $f \in \mathcal{H}(\Omega)$. Show that if f is real-valued in Ω , then f is constant.

3. Suppose that Ω is a region and $f \in \mathcal{H}(\Omega)$. Suppose that $z \mapsto |f(z)|$ is constant on Ω . Show that f must be constant. **Hint:** Consider $|f(z)|^2$.

We let f, u, v and Ω be as above. Define

$$F : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{by} \quad F(x, y) = (u(x, y), v(x, y)).$$

Recall that F is differentiable at $(x_0, y_0) \in \Omega$ if there is a linear function $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\lim_{(h,k) \rightarrow (0,0)} \frac{\|F(x_0 + h, y_0 + k) - F(x_0, y_0) - L(h, k)\|}{\|(h, k)\|} = 0,$$

in which case, the partials of u and v must exist and L is given by the Jacobian Matrix

$$[L] = \begin{pmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{pmatrix}.$$

(Here $\|(x, y)\| = \sqrt{x^2 + y^2} = |x + iy|$.)

4. Let f, F, u, v and Ω be as above. Let $z_0 = x_0 + iy_0 \in \Omega$. Show that $f'(z_0)$ exists if and only if the Cauchy-Riemann equations hold at z_0 and F is differentiable at (x_0, y_0) .

Hint: if we let $z = h + ik$ and if T is given by the matrix

$$[T] = \begin{pmatrix} u_x(x_0, y_0) & -v_x(x_0, y_0) \\ v_x(x_0, y_0) & u_x(x_0, y_0) \end{pmatrix},$$

then

$$\|F(x_0 + h, y_0 + k) - F(x_0, y_0) - T(h, k)\| = |f(z + z_0) - f(z_0) - \omega z|,$$

where $\omega = u_x(x_0, y_0) + iv_x(x_0, y_0) = f_x(z_0)$.

Question #4 has an important corollary. We learn in multivariable calculus, that F is differentiable at (x_0, y_0) if the partial derivatives of u and v exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0) . Hence we get as a corollary of Question #4, with f, u and v defined as above, that if u and v have continuous partial derivatives in a neighborhood of (x_0, y_0) and if the Cauchy-Riemann equations hold at z_0 , then $f'(z_0)$ exists. Use this observation in Question #5.

5. Define $\exp : \mathbb{C} \rightarrow \mathbb{C}$ by $\exp(x + iy) = e^x(\cos(y) + i\sin(y))$. Show that $\exp \in H(\mathbb{C})$ and $\exp'(z) = \exp(z)$ for all $z \in \mathbb{C}$.

If Ω is open in \mathbb{C} or \mathbb{R}^2 , then we say $u : \Omega \rightarrow \mathbb{R}$ is *harmonic* if it has continuous second partial derivatives and if it is a solution to Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{L}$$

6. Suppose that $f \in \mathcal{H}(\Omega)$. Let $u(x, y) = \operatorname{Re}(f(x + iy))$. Assuming u has continuous second partials, show that u is harmonic in Ω .

7. Suppose that $u : \Omega \rightarrow \mathbb{R}$ is harmonic. We say $v : \Omega \rightarrow \mathbb{R}$ is a *harmonic conjugate* for u if $f(x + iy) = u(x, y) + iv(x, y)$ defines a holomorphic function on Ω . Find all harmonic conjugates for $u(x, y) = 2xy$.

For the purposes of this assignment only, we'll call a region Ω a *SC region* if every $f \in \mathcal{H}(\Omega)$ has an antiderivative in Ω . For example, we have shown in lecture that every convex region is a SC region. Later, I hope that we'll see that any simply connected region is a SC region. In fact, a region is a SC-region if and only if it is simply connected.

8. Suppose that Ω is a SC region and that u is harmonic in Ω . Show that u has a harmonic conjugate in Ω .

Hint: we need to find a function $f \in \mathcal{H}(\Omega)$ such that $u = \operatorname{Re}(f)$. However, consider $g = u_x - iu_y$. Show that $g \in \mathcal{H}(\Omega)$ and consider an anti-derivative f for g in Ω . As in Question 1, you may use without proof the fact that if $w : \Omega \rightarrow \mathbb{R}$ is continuous and $w_x \equiv 0 \equiv w_y$ in Ω , then w is constant.

If $u = \operatorname{Re}(f)$, then $u_x = \operatorname{Re}(f')$ and $u_y = \operatorname{Re}(-if')$. Thus, it is a consequence of Question #8 (and the deep result that $f \in \mathcal{H}(\Omega)$ implies f is analytic) that every harmonic function has continuous partial derivatives of all orders.

9. Just as in Question #5, we shall write $\exp(z)$ in place of e^z . Suppose that Ω is a SC region and that $0 \notin \Omega$. Then show there is a $f \in \mathcal{H}(\Omega)$ such that

$$\exp(f(z)) = z.$$

We call f a *branch of $\log(z)$ in Ω* . (Hint: start by letting f be an antiderivative of $1/z$. and recall that $\exp(z) = a$ has infinitely many solutions for all $a \neq 0$.)

10. Show that $f(z) = 1/z$ can't have an antiderivative in the punctured complex plane $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Conclude that there is no (holomorphic) branch of $\log z$ in \mathbb{C}^* .