Math 73/103: Measure Theory and Complex Analysis
Fall 2019 - Homework 3

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1. Suppose that ρ is a premeasure on an algebra $\mathcal{A}$ of sets in X. Let ρ^o be the associated outer measure.

(a) Show that $\rho^o(E) = \rho(E)$ for all $E \in \mathcal{A}$.

(b) If $\mathcal{M}^o$ is the σ-algebra of ρ^o-measurable sets, show that $\mathcal{A} \subset \mathcal{M}^o$.

2. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Recall that $E \in \mathcal{M}$ is called σ-finite if E is the countable union of sets of finite measure. Let $f \in \mathcal{L}^1(\mu)$.

(a) Show that $\{\, x \in X : f(x) \neq 0 \,\}$ is σ-finite.

(b) Suppose that $f \geq 0$. Explain why there are measurable simple functions φ_n such that $\varphi_n \nearrow f$ almost everywhere and there is a single σ-finite set outside of which the φ_n vanish.

(c) Given $\varepsilon > 0$ show that there is a simple function such that $\displaystyle\int_X |f - \varphi| \, d\mu < \varepsilon$.

(d) If $(X, \mathcal{M}, \mu) = (\mathbb{R}, \Lambda^o, \lambda)$ is Lebesgue measure, show that we can take the simple function φ in part (c) to be a step function — that is, a finite linear combination of characteristic functions of *intervals*.

3. Suppose that $f \in \mathcal{L}^1(\mathbb{R}, \Lambda^o, \lambda)$ is a Lebesgue integrable function on the real line. Let $\varepsilon > 0$. Show that there is a continuous function g that vanishes outside a bounded interval such that $\|f - g\|_1 < \varepsilon$.
Hint: this is easy if f is the characteristic function of a bounded interval: draw a picture.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(f_n)_n, f$ functions, such that $f, f_n : X \to \mathbb{C}$ measurable. Show that if $f_n \xrightarrow{\mu} f$ then there is a subsequence $(f_{n_k})_k$ such that $f_{n_k} \to f$ almost everywhere.
Hint: for each k let n_k be such that

$$n \geq n_k \Rightarrow \mu\left(\left\{x \in X : |f_n(x) - f(x)| \geq 2^{-k}\right\}\right) < 2^{-k}.$$

One of Littlewood's Principles is that a pointwise convergent sequence of functions is nearly uniformly convergent. This is also known as "Egoroff's Theorem".

5. Prove Egoroff's Theorem: Suppose that $(X, \mathcal{M}, \mu)$ is a finite measure space that is, $\mu(X) < \infty$. Suppose that $\{f_n\}$ is a sequence of measurable functions converging almost everywhere to a measurable function $f : X \to \mathbb{C}$. Then for all $\varepsilon > 0$ there is a set $E \in \mathcal{M}$ such that $\mu(X \setminus E) < \varepsilon$ and $f_n \to f$ uniformly on E.

Some suggestions:

(a) There is no harm in assuming that $f_n \to f$ everywhere.

(b) Let $E_n(k) = \bigcup_{m=n}^{\infty} \{ x \in X : |f_m(x) - f(x)| \geq \frac{1}{k} \}$.

(c) Show that $\lim_{n\to\infty} \mu(E_n(k)) = 0$. (You need $\mu(X) < \infty$ here.)

(d) Fix $\varepsilon > 0$ and k. Choose $n_k \geq n$ so that $\mu(E_{n_k}(k)) < \frac{\varepsilon}{2^k}$, and let $E = \displaystyle\bigcup_{k=1}^{\infty} E_{n_k}(k)$.