Math 73/103: Measure Theory and Complex Analysis
Fall 2019 - Homework 1

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1. Show that the *countable* union of sets of measure zero in $\mathbb{R}$ has measure zero.

2. Suppose $f : [a,b] \to \mathbb{R}$ is bounded, and let $\mathcal{P}$ and $\mathcal{Q}$ be subdivisions of $[a,b]$. Prove that $L(f,\mathcal{P}) \le U(f,\mathcal{Q})$, where $L(f,\mathcal{P})$ and $U(f,\mathcal{Q})$ are the lower and upper Riemann sums, respectively, for f on $[a,b]$.
Hint: The result is trivial if $\mathcal{P} = \mathcal{Q}$; now let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$.

3. Prove that a bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if and only if for all $\varepsilon > 0$ there is a subdivision $\mathcal{P}$ of $[a,b]$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon.$$

4. (*Rudin*: page 31 #1) Suppose that $(X,\mathcal{M})$ is a measurable space. Show that if $\mathcal{M}$ is countable, then $\mathcal{M}$ is finite.
Hint: Since $\mathcal{M}$ is countable, you can show that $\omega_x = \bigcap\{\, E : E \in \mathcal{M} \text{ and } x \in E \,\}$ belongs to $\mathcal{M}$. The sets $\{\, \omega_x \,\}_{x\in X}$ partition X.

5. Let X be an uncountable set and let $\mathcal{M}$ be the collection of subsets E of X such that either E or E^c is countable. Prove that $\mathcal{M}$ is a σ-algebra.

6. Recall from calculus that if $\{\, a_n \,\}$ is a sequence of nonnegative real numbers, then $\sum_{n=1}^{\infty} a_n = \sup_n s_n$, where $s_n = a_1 + \cdots + a_n$. (Note the value ∞ is allowed.)

   (a) Show that $\sum_{n=1}^{\infty} a_n = \sup\{\, \sum_{k\in F} a_k : F \text{ is a finite subset of } \mathbb{Z}^+ = 1,2,3,\dots \,\}$.
   Note: The point of this problem is that if I is a (not necessarily countable) set, and if $a_i \ge 0$ for all $i \in I$, then we can define $\sum_{i\in I} a_i = \sup\{\, \sum_{k\in F} a_k : F \text{ is a finite subset of } I \,\}$, and our new definition coincides with the usual one when both make sense.

   (b) Now let X be a set and $f : X \to [0,\infty)$ a function. For each $E \subset X$, define

$$\nu(E) := \sum_{x\in E} f(x).$$

   Show that ν is a measure on $\big(X, \mathcal{P}(X)\big)$. In lecture, we considered the special cases of *counting measure*, where $f(x) = 1$ for all $x \in X$, and the *delta measure at* x_0, where $f(x_0) = 1$ for some $x_0 \in X$ and $f(x) = 0$ otherwise. Another important example is the case where $\sum_{x\in X} f(x) = 1$. Then f is a (discrete) probability distribution on X and $\nu(E)$ is the probability of the event E for this distribution.

(c) Let X, f, and ν be as in part (b). Show that if $\nu(E) < \infty$, then $\{\, x \in E : f(x) > 0 \,\}$ is countable.
Hint: If $\{\, x \in E : f(x) > 0 \,\}$ is uncountable, then for some $m \in \mathbb{Z}^+$, the set

$$\{\, x \in E : f(x) > \frac{1}{m} \,\} \text{ is infinite.}$$

This last result says that discrete probability distributions "live on" countable sample spaces.

7. (*Rudin*: page 31 #3) Prove that if f is a real-valued function on a measurable space $(X, \mathcal{M})$ such that $\{\, x : f(x) \geq r \,\}$ is measurable for all rational r, then f is measurable.

8. (*Rudin*: page 31 #5) Suppose that $f, g : (X, \mathcal{M}) \to [-\infty, \infty]$ are measurable functions. Prove that the sets

$$\{\, x : f(x) < g(x) \,\} \quad \text{and} \quad \{\, x : f(x) = g(x) \,\}$$

are measurable.
Remark: If $h = f - g$ were defined, then this problem would be much easier (why?). The problem is that $\infty - \infty$ and $-\infty + \infty$ make no sense, so h may not be everywhere defined.