Math 103: Measure Theory and Complex Analysis
Fall 2018

Prerequisites

1 Set Theory

We recall the basic facts about countable and uncountable sets, union and intersection of sets and images and preimages of functions.

1.1 Countable and uncountable sets

We can compare infinite sets via bijections or one-to-one correspondences.

Definition 1 Let I be an arbitrary set.

a) The set I is **finite**, if there is a bijective map $f : I \to \{1, 2, 3, \ldots, n\}$ for some positive integer n.

b) The set I is **infinite**, if it is not finite.

c) The set I is **countably infinite**, if there is a bijective map $f : I \to \mathbb{N}$.

d) The set I is **countable**, if it is either finite or countably infinite.

e) The set I is **uncountable**, if it is not countable.

We recall:
Theorem 2 A subset of a countably infinite set is countable.

We have furthermore the important theorem:
Theorem 3 A countable union of countable sets is countable.

proof: It is sufficient to prove the statement for a disjoint union $A = \biguplus_{i=1}^{\infty} A_i$ of countably infinte sets A_i. This is true as

1.) Each union of sets $\bigcup_{i=1}^{\infty} B_i$ can be decomposed into a disjoint union $\biguplus_{i=1}^{\infty} B_i'$ of sets by removing multiple occurences.

2.) Each finite set B_i' can be extended to an infinite set $\tilde{B}_i$, such that $\tilde{B}_i \cap B_k' = \emptyset$ for all $k \neq i$.

3.) If $\biguplus_{i=1}^{\infty} \tilde{B}_i$ is countably infinite, then the subset $\bigcup_{i=1}^{\infty} B_i$ is countable by **Theorem 2**.

So suppose we have a disjoint union $\biguplus_{i=1}^{\infty} A_i$ of countably infinte sets A_i. We list all elements of $A = \biguplus_{i=1}^{\infty} A_i$:

$$\begin{aligned} A_1 &= \{x_{11}, x_{12}, x_{13}, \ldots, x_{1n}, \ldots\} \\ A_2 &= \{x_{21}, x_{22}, x_{23}, \ldots, x_{2n}, \ldots\} \\ &\vdots \\ A_m &= \{x_{m1}, x_{m2}, x_{m3}, \ldots, x_{mn}, \ldots\} \\ &\vdots \end{aligned}$$

As the factorization into primes is unique, we know that the set of positive integers $S = \{2^k \cdot 3^n, n, k \in \mathbb{N}\}$ satisfies:

$$2^{k_1} \cdot 3^{n_1} = 2^{k_2} \cdot 3^{n_2} \Leftrightarrow k_1 = k_2 \text{ and } n_1 = n_2. \quad (*)$$

Hence the assignment $f : S \to \biguplus_{i=1}^{\infty} A_i$, defined by $f(2^k \cdot 3^n) = x_{kn}$ is a well-defined map which is bijective. Hence $\biguplus_{i=1}^{\infty} A_i$ is in one-to-one correspondence with a subset of $\mathbb{N}$, which by **Theorem 2** is countable. Hence $A = \biguplus_{i=1}^{\infty} A_i$ is also countable. $\square$

Examples 4 $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countable, hence by the previous theorem we know that

$$\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \biguplus_{i \in \mathbb{Z}} (i, \mathbb{Z}) \quad \text{and} \quad \mathbb{Q}^2 = \mathbb{Q} \times \mathbb{Q} = \biguplus_{q \in \mathbb{Q}} (q, \mathbb{Q})$$

are countable. Using this argument iteratively we have that for fixed n, $\mathbb{Z}^n$ and $\mathbb{Q}^n$ are countable.

1.2 Sets and functions

Theorem 1 (De Morgan's Law) Let $(A_i)_{i \in I} \subset X$ be a collection of sets in X. If $A^c = X \backslash A$ for all $A \subset X$, then

a) $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$

b) $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$

proof a) We have

$$\begin{aligned} \left(\bigcup_{i \in I} A_i\right)^c &= X \backslash \{x \in X \mid \exists i \in I, \text{ such that } x \in A_i\} = \\ &\{x \in X \mid \neg(\exists i \in I, \text{ such that } x \in A_i)\} = \\ &\{x \in X \mid \forall i \in I, x \notin A_i\} = \{x \in X \mid \forall i \in I, x \in A_i^c\} = \bigcap_{i \in I} A_i^c. \ \square \end{aligned}$$

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b) Similarly

$$\begin{aligned}\left(\bigcap_{i\in I} A_i\right)^c &= X\backslash\{x\in X \mid \forall i\in I, x\in A_i\} = \\ &\{x\in X \mid \neg(\forall i\in I,\ \text{we have that } x\in A_i)\} = \\ &\{x\in X \mid \exists i\in I,\ \text{such that } x\notin A_i\} = \{x\in X \mid \exists i\in I, x\in A_i^c\} = \bigcup_{i\in I} A_i^c.\square\end{aligned}$$

Lemma 2 (Functions and Sets) Let $f: X\to Y$ be a function. Let $(A_j)_{j\in J}\subset X$ be a collection of sets in X. Let furthermore $(B_i)_{i\in I}\subset Y$ be a collection of sets in Y and $B\subset Y$. Then

a) $f\left(\bigcap_{j\in J} A_j\right)\subset \bigcap_{j\in J} f(A_j)$.

b) $\bigcup_{j\in J} f(A_j) = f\left(\bigcup_{j\in J} A_j\right)$.

c) $f^{-1}(B)^c = f^{-1}(B^c)$.

d) $\bigcup_{i\in I} f^{-1}(B_i) = f^{-1}\left(\bigcup_{i\in I} B_i\right)$ and $\bigcap_{i\in I} f^{-1}(B_i) = f^{-1}\left(\bigcap_{i\in I} B_i\right)$.

proof a) We have that

$$\begin{aligned}f\left(\bigcap_{j\in J} A_j\right) &= \{y\in Y \mid y=f(x) \text{ and } f(x)\in f(\bigcap_{j\in J} A_j)\} = \\ &\{f(x)\in Y \mid \forall j\in J,\ \text{we have } x\in A_j\}.\end{aligned}$$

Furthermore

$$\begin{aligned}\bigcap_{j\in J} f(A_j) &= \{y\in Y \mid \forall j\in J,\ \text{we have } y\in f(A_j)\} = \\ &\{f(x)\in Y \mid \forall j\in J,\ \text{we have } f(x)\in f(A_j)\}.\end{aligned}$$

Now if $x\in A_j$ then $f(x)\in f(A_j)$ and the first set is contained in the second. $\square$

Note We see that the converse is not true by taking $f:\{1,2\}\to\{1\}$, where $f(1)=f(2)=1$. For $A_1=\{1\}$ and $A_2=\{2\}$ we get

$$f(A_1\cap A_2)=\emptyset, \quad \text{but} \quad f(A_1)\cap f(A_2)=\{1\}.$$

b) We know that

$$\begin{aligned}\bigcup_{j\in J} f(A_j) &= \{y\in Y \mid \exists j\in J,\ \text{such that } y\in f(A_j)\} = \\ &\{f(x)\in Y \mid \exists j\in J,\ \text{such that } f(x)\in f(A_j)\}.\end{aligned}$$

On the other hand

$$f\left(\bigcup_{j\in J} A_j\right) = \{y \in Y \mid y = f(x) \text{ and } \exists j \in J, \text{ such that } x \in A_j\} =$$
$$\{f(x) \in Y \mid \exists j \in J, \text{ such that } x \in A_j\}.$$

But if $x \in A_j$ then $f(x) \in f(A_j)$ and the second set is contained in the first. On the other hand if $f(x) \in f(A_j)$ for some $j \in J$ then there is x', such that $f(x) = f(x')$ and $x' \in A_j$ for some $j \in J$. So the first set is contained in the second. □

c) We know that

$$\bigcup_{i\in I} f^{-1}(B_i) = \{x \in X \mid \exists i \in I, \text{ such that } x \in f^{-1}(B_i)\} = \{x \in X \mid \exists i \in I, \text{ such that } f(x) \in B_i\}.$$

We compare this with

$$f^{-1}\left(\bigcup_{i\in I} B_i\right) = \{x \in X \mid f(x) \in \bigcup_{i\in I} B_i\} = \{x \in X \mid \exists i \in I, \text{ such that } f(x) \in B_i\}$$

which shows that the sets are equal. We prove the second statement in a similar fashion. □

d) We know that

$$f^{-1}(B)^c = X \backslash f^{-1}(B) = X \backslash \{x \in X \mid f(x) \in B\} = \{x \in X \mid f(x) \notin B\}.$$

On the other hand we have that

$$f^{-1}(B^c) = \{x \in X \mid f(x) \in B^c\} = \{x \in X \mid f(x) \notin B\}$$

and the two sets are equal.

2 Topology

The proofs of the following theorems can be found in Munkres, *Topology, 2nd edition*, **Chapter 2**, **Section 12,13** and **20**.

2.1 Basics

Definition 1 Let X be a set. A **topology** on X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X, such that

a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

b) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$ ($\mathcal{T}$ is closed under intersection).

c) $(A_k)_{k\in K} \subset \mathcal{T} \Rightarrow \bigcup_{k\in K} A_k \in \mathcal{T}$ ($\mathcal{T}$ is closed under **any** union).

In this case the elements of $\mathcal{T}$ the **open subsets** of X and $(X, \mathcal{T})$ is called a **topological space**.

Examples $\mathcal{T} = \{\emptyset, X\}$ or $\mathcal{T}' = \mathcal{P}(X)$.

Remark 2 b) implies that $\mathcal{T}$ is stable under finite intersections.

Definition 3 Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be topological spaces. A function $f : X \to X'$ is **continuous** if

$$f^{-1}(A') \in \mathcal{T} \quad \text{for all} \quad A' \in \mathcal{T}'.$$

Definition 4 (Basis) Let $(X, \mathcal{T})$ be a topological space. Then $\beta \subset \mathcal{T}$ is a basis for the topology $\mathcal{T}$ if

$$\text{for all } \ A \in \mathcal{T} \ \text{ we have that } \ A = \bigcup_{i\in I} A_i \ \text{ where } \ (A_i)_{i\in I} \subset \beta.$$

This means that every element in $\mathcal{T}$ is a union of elements of β.

Theorem 5 (Basis = neighbourhood basis) β is a basis for the topology $\mathcal{T}$ iff

$$\text{for all } A \in \mathcal{T} \text{ and for all } x \in A \ \exists\ U(x) = U \in \beta, \ \text{ such that } \ x \in U \subset A.$$

Definition 6 (second countable) A topological space $(X, \mathcal{T})$ is called **second countable** if there is a countable basis for its topology.

Example A second countable basis for the usual topology of the real line $\mathbb{R}$ is given by the intervals with rational endpoints.

Proposition 7 If (X, d) is a metric space with a countable dense subset, the topology induced by the metric is second countable.

proof We know that

1.) the basis β_d of the topology $\mathcal{T}_d$ induced by the metric d is the collection of open balls in (X, d): $\beta_d = \{B_r(x) \mid r \in \mathbb{R}^+, x \in X\}$

2.) there is a countable dense subset $D = (x_n)_{n\in\mathbb{N}} \subset X$ in X.

3.) by **Theorem 5**, as β_d is a basis, we know that for all $A \in \mathcal{T}_d$ and $x \in A$ there is $B_r(x') \subset \beta_d$, such that $x \in B_r(x') \subset A$.

We take

$$\beta = \{B_{\frac{1}{m}}(x_n) \mid m, n \in \mathbb{N}\}.$$

Take $A \in \mathcal{T}$ and $x \in A$ as in 3.). From this condition it follows that it is sufficient to show that there is a ball $B_{\frac{1}{m}}(x_n) \in \beta$, such that $B_{\frac{1}{m}}(x_n) \subset B_r(x')$. Furthermore, if $x' \neq x$, we can find a ball of smaller radius around x that also satisfies 3.). Hence we can assume that $x' = x$.
To construct our ball we take $m \in \mathbb{N}$, such that $\frac{r}{2} > \frac{1}{m} \Leftrightarrow r > \frac{2}{m}$. By the density of D there is $x_n \in D$, such that $d(x_n, x) < \frac{1}{m}$. Then for every point $\tilde{x} \in B_{\frac{1}{m}}(x_n)$ we have by the triangle inequality:

$$d(\tilde{x}, x) \leq d(\tilde{x}, x_n) + d(x_n, x) < \frac{1}{m} + \frac{1}{m} < r$$

Hence $x \in B_{\frac{1}{m}}(x_n) \subset B_r(x) \subset A$ and therefore β is a countable basis for $\mathcal{T}$. □

3 Limits

We recall the defintion of infimum and supremum and lim inf and lim sup. The correspondig theorems and definitions can be, for example found in *Gordon, Real Analysis - A First Course, 2nd edition.*

3.1 Infimum and supremum

Definition 1 Let $S \subset \mathbb{R}$ be a non-empty set of real numbers. Suppose S is bounded above. The number β is the **supremum of** S if β is an upper bound of S and any number less than β is not an upper bound of S i.e.

$$\text{for all } \ b < \beta \ \text{ there is an } \ x \in S, \ \text{ such that } \ b < x.$$

We will write $\beta = \sup(S)$.

Definition 2 Let $S \subset \mathbb{R}$ be a non-empty set of real numbers. Suppose S is bounded below. The number α is the **infimum of** S if α is a lower bound of S and any number greater than α is not a lower bound of S i.e.

$$\text{for all } \ a > \alpha \ \text{ there is an } \ x \in S, \ \text{ such that } \ a > x.$$

We will write $\alpha = \inf(S)$.

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3.2 The extended real number line

see Wilkins: *The extended real number system.*

3.3 Limit superior and limit inferior

We recall the following definitions from real analysis:

Let $(a_n)_{n\in\mathbb{N}} \subset \bar{\mathbb{R}}$ be a sequence. For $k \geq 1$ consider the new sequence

$$b_k = \sup_{n\geq k} a_n = \sup\{a_k, a_{k+1}, a_{k+2}, a_{k+3}, \ldots\}$$

Then $b_k \geq b_{k+1}$ for all $k \in \mathbb{N}$ and therefore $\lim_{k\to\infty} b_k = \inf_{k\in\mathbb{N}} b_k \in \bar{\mathbb{R}}$. We define:

Definition 1 (Limit superior and inferior) We call the **limit superior** of a sequence $(a_n)_n \subset \bar{\mathbb{R}}$ the number

$$\limsup_{n\in\mathbb{N}} a_n \overset{\textbf{Def.}}{=} \lim_{k\to\infty} b_k = \inf_{k\in\mathbb{N}} b_k.$$

In a similar fashion we call the **limit inferior** of a sequence $(a_n)_n \subset \bar{\mathbb{R}}$ the number

$$\liminf_{n\in\mathbb{N}} a_n \overset{\textbf{Def.}}{=} \lim_{k\to\infty} \inf_{n\geq k} a_n.$$

Example The sequence $(a_n)_{n\in\mathbb{N}} = \left(\frac{\cos(n)}{n}\right)_{n\in\mathbb{N}}$ and the sequence $(c_k)_{k\in\mathbb{N}}$ where $c_k = \inf_{n\geq k} a_n$.

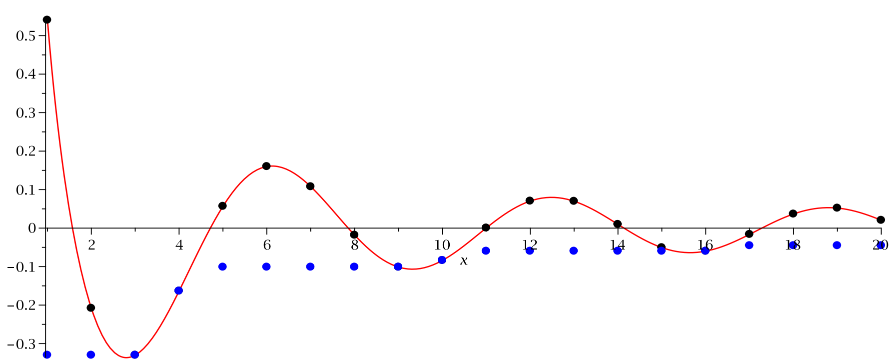

Figure 1: Plot of $\frac{cos(x)}{x}$ (red) and the sequence $(a_n)_{n\in\mathbb{N}} = \left(\frac{\cos(n)}{n}\right)_{n\in\mathbb{N}}$ (black) and the sequence given by $c_k = \inf_{n\geq k} a_n$ (blue).

Proposition 2 For a sequence $(a_n)_{n\in\mathbb{N}} \subset \bar{\mathbb{R}}$ we have that

a) $\liminf_{n\in\mathbb{N}} a_n \leq \limsup_{n\in\mathbb{N}} a_n$.

b) $\lim_{n\to\infty} a_n$ exists if and only if $\liminf_{n\in\mathbb{N}} a_n = \lim_{n\to\infty} a_n = \limsup_{n\in\mathbb{N}} a_n$.

4 Complex analysis

see Beck et al.: *A first course in complex analysis.*