Math 103: Measure Theory and Complex Analysis
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Lecture 6

As a consequence of the **MCT** we have:
Theorem 7 Let $(f_n)_{n\in\mathbb{N}} : (X,\mathcal{M}) \to [0,\infty]$ be a sequence of measurable functions on X such that

$$f(x) = \sum_{n\in\mathbb{N}} f_n(x) \text{ for all } x \in X.$$

Then f is measurable and $\displaystyle\sum_{n=1}^{\infty}\int_X f_n\,d\mu = \int_X f\,d\mu = \int_X \sum_{n=1}^{\infty} f_n\,d\mu$.

proof Idea: Setting $g_N := \sum_{n=1}^N f_n(x)$, the sequence $(g_N)_N$ satisfies the conditions of the previous theorem. This means that

$$\int_X \sum_{n=1}^{\infty} f_n\,d\mu = \lim_{N\to\infty}\int_X g_N\,d\mu \overset{\text{MCT}}{=} \int_X f\,d\mu. \tag{1}$$

It remains to show that the integral and the sum can be exchanged. We have to argue again with simple functions. By **Ch. 1.4. Theorem 3** we know that there are increasing sequences of simple measurable functions $(s_i^1)_i$ and $(s_i^2)_i$, such that

$$\lim_{i\to\infty} s_i^1 = f_1 \text{ and } \lim_{i\to\infty} s_i^2 = f_2$$

As they satisfy the conditons of the MCT, we know that

$$\lim_{i\to\infty}\int_X s_i^1\,d\mu \overset{\text{MCT}}{=} \int_X f_1\,d\mu \text{ and } \lim_{i\to\infty}\int_X s_i^2\,d\mu \overset{\text{MCT}}{=} \int_X f_2\,d\mu.$$

Then

$$\lim_{i\to\infty} s_i^1 + s_i^2 = f_1 + f_2.$$

and by taking the sequence $(s_i^1 + s_i^2)_i$ again by the MCT

$$\int_X f_1+f_2\,d\mu \overset{\text{MCT}}{=} \lim_{i\to\infty}\int_X s_i^1+s_i^2\,d\mu \overset{\textbf{Prop. 5}}{=} \lim_{i\to\infty}\int_X s_i^1\,d\mu+\lim_{i\to\infty}\int_X s_i^2\,d\mu \overset{\text{MCT}}{=} \int_X f_1\,d\mu+\int_X f_2\,d\mu.$$

Here the inequality in the middle is true, as $\int_X s_i^1 + s_i^2\,d\mu = \int_X s_i^1\,d\mu + \int_X s_i^2\,d\mu$ for all i. In total we have that

$$\int_X f_1 + f_2\,d\mu = \int_X f_1\,d\mu + \int_X f_2\,d\mu.$$

It follows by induction that $\int_X \sum_{n=1}^N f_n\,d\mu = \sum_{n=1}^N \int_X f_n\,d\mu$ for any $N \in \mathbb{N}$. Then with Equation (1) we have that

$$\int_X f\,d\mu \overset{\text{MCT}}{=} \lim_{N\to\infty}\int_X g_N\,d\mu = \lim_{N\to\infty}\int_X \sum_{n=1}^N f_n\,d\mu = \lim_{N\to\infty}\sum_{n=1}^N\int_X f_n\,d\mu = \sum_{n=1}^{\infty}\int_X f_n\,d\mu \quad \square$$

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Corollary 8 Let $(a_{ij})_{i,j\in\mathbb{N}} \subset \mathbb{R}_0^+$ be a countable subset of $\mathbb{R}$. Taking the counting measure we obtain
$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$
Using the MCT we can also prove:

Theorem 9 (Fatou's lemma) Let $(f_n)_{n\in\mathbb{N}} : (X, \mathcal{M}) \to [0,\infty]$ be a sequence of measurable functions on X. Then
$$\int_X \liminf_{n\in\mathbb{N}} f_n \, d\mu \le \liminf_{n\in\mathbb{N}} \int_X f_n \, d\mu.$$
proof We set for all $x \in X$ and $k \in \mathbb{N}$
$$\boxed{g_k(x) = \inf_{n\ge k} f_n(x).}$$
It follows that

- By **Ch. 1.3 Theorem 4** g_k is measurable.
- $0 \le g_1 \le g_2 \le \ldots$ i.e $(g_k)_k$ is an increasing sequence.
- $\lim\limits_{k\to\infty} g_k = \liminf\limits_{n\in\mathbb{N}} f_n$

Hence the conditions of the MCT are satisfied and we conclude
$$\boxed{\lim_{k\to\infty}\int_X g_k \, d\mu = \int_X \liminf_{n\in\mathbb{N}} f_n \, d\mu}.$$
Furthermore for all $k \in \mathbb{N}$ we have that
$$g_k \le f_k \Rightarrow \int_X g_k \, d\mu \le \int_X f_k \, d\mu \Rightarrow \liminf_k \int_X g_k \, d\mu \le \liminf_k \int_X f_k \, d\mu. \quad (*)$$
But as the limit exists we have that $\liminf_k \int_X g_k \, d\mu = \lim_{k\to\infty}\int_X g_k \, d\mu = \int_X \liminf_n f_n \, d\mu$. Therefore
$$\int_X \liminf_n f_n \, d\mu = \lim_{k\to\infty}\int_X g_k \, d\mu \overset{(*)}{\le} \liminf_n \int_X f_n \, d\mu. \quad \square$$
Note 9 The inequality can be strict: Let $E, F \in \mathcal{M}$, such that $E \cap F = \emptyset$ and $\mu(E) > 0, \mu(F) > 0$. Set
$$f_n = \begin{cases} \mathbb{1}_E & \text{if} \quad n \text{ even} \\ \mathbb{1}_F & \quad\quad n \text{ odd} \end{cases}.$$
Then $\liminf_{n\in\mathbb{N}} f_n = 0$ but $\liminf_{n\in\mathbb{N}} \int_X f_n \, d\mu = \min\{\mu(E), \mu(F)\} > 0$.

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We can also obtain new measures by integrating positive functions.

Theorem 10 Let $f : (X, \mathcal{M}) \to [0, \infty]$ be a measurable function on X. Then

$$\varphi(E) = \int_E f \, d\mu$$

defines a measure on $(X, \mathcal{M})$. Furthermore for any measurable function $g : (X, \mathcal{M}) \to [0, \infty]$ we have $\displaystyle\int_X g \, d\varphi = \int_X gf \, d\mu$.

proof Idea for the first part: We use **Theorem 7** which is a consequence of the MCT. To this end we observe that $\int_B f d\mu = \int_X f \cdot \mathbb{1}_B d\mu$.
Clearly $\varphi(\emptyset) = 0 < \infty$, so the first condition for a measure is fulfilled. It remains to show that φ is countably additive. Let $(B_k)_{k\in\mathbb{N}}$ be a collection of mutually disjoint elements of $\mathcal{M}$ and $B = \biguplus_{k\in\mathbb{N}} B_k$, then

$$\varphi(B) = \int_B f d\mu = \int_X f \cdot \mathbb{1}_B \; d\mu \overset{B=\biguplus_{k\in\mathbb{N}} B_k}{=} \int_X \sum_{k=1}^{\infty} f \cdot \mathbb{1}_{B_k} \; d\mu \overset{\textbf{Thm. 7}}{=}$$
$$\sum_{k=1}^{\infty} \int_X f \cdot \mathbb{1}_{B_k} \; d\mu = \sum_{k=1}^{\infty} \int_{B_k} f \, d\mu = \sum_{k=1}^{\infty} \varphi(B_k).$$

Idea for the second part: The statement is true for nnsfs, then it is true for functions.
We start with the simple case where $g = \mathbb{1}_A$ for $A \in \mathcal{M}$. In this case we verify:

$$\int_X \mathbb{1}_A \; d\varphi \overset{\text{Def. } \int}{=} \varphi(A) \overset{\text{Def. } \varphi}{=} \int_A f \, d\mu = \int_X f \cdot \mathbb{1}_A \; d\mu.$$

By linearity the result is also true for nnsfs (*). Finally, a function g as in the theorem can be approximated by an increasing sequence of nnnsfs $(s_k)_k$, such that $\lim_{k\to\infty} s_k = g$. We conclude

$$\int_X g \, d\varphi \overset{MCT}{=} \lim_{k\to\infty} \int_X s_k \, d\varphi \overset{(*)}{=} \lim_{k\to\infty} \int_X s_k \cdot f \, d\mu \overset{MCT}{=} \int_X g \cdot f \, d\mu.$$

Here the last equation follows as the sequence $(s_k \cdot f)_k$ approximates $g \cdot f$. □

Remark Note that $\mu(E) = 0$ implies that $\varphi(E) = 0$ (recall that $0 \cdot \infty = 0$). In this case we say that φ is **absolutely continuous with respect to** μ and write $\varphi \ll \mu$.
Under this condition the converse of the theorem is also true, i.e. there is a function f, such that for all $E \in \mathcal{M}$

$$\varphi(E) = \int_E f \, d\mu.$$

This is the **Radon-Nikodym Theorem** which we will prove later.

Ch. 1.7 Integration of complex functions

Outline: We extend integration to complex functions. The key observation is that every complex function $f = u + i \cdot v$ can be written as

$$f = (u^+ - u^-) + i \cdot (v^+ - v^-)$$

where $u^+(x) = \max\{0, u(x)\}$ and $u^-(x) = -\min\{0, u(x)\}$. Therefore all functions on the right hand side are real valued and positive.

Definition 1 (Lebesgue integrable functions) Let $(X, \mathcal{M}, \mu)$ be a measure space. We call

$$\mathcal{L}^1(\mu) = \{f : (X, \mathcal{M}) \to \mathbb{C} \mid f \text{ measurable }, \int_X |f| \, d\mu < \infty\}$$

the set of **Lebesgue integrable functions**.

Note 2 By **Ch.1.3 Corollary 17** we have that $f = u + i \cdot v$ measurable $\Rightarrow |f|$ measurable.

We still have not defined integration, in a natural way we set

Defintion 3 (Integration) If $f = u + i \cdot v \in \mathcal{L}^1(\mu)$ and $E \in \mathcal{M}$, then

$$\int_E f \, d\mu = \int_E u^+ \, d\mu - \int_E u^- \, d\mu + i \cdot \left(\int_E v^+ \, d\mu - \int_E v^- \, d\mu \right). \tag{2}$$

If $g : (X, \mathcal{M}) \to \bar{\mathbb{R}}$ is measurable and $\int_E g^+ \, d\mu < \infty$ or $\int_E g^- \, d\mu < \infty$, then we set

$$\int_E g \, d\mu = \int_E g^+ \, d\mu - \int_E g^- \, d\mu.$$

Note 4 All integrals in (2) are finite. We have, for example:

$$u^+ \leq |u| \leq |f| \Rightarrow \int_E u^+ \, d\mu \leq \int_E |f| \, d\mu \leq \infty.$$

We now prove that $\mathcal{L}^1(\mu)$ is a vector space. To this end it is sufficient to show that $\mathcal{L}^1(\mu)$ is a subspace of the vector space of complex valued functions. Clearly $0 \in \mathcal{L}^1(\mu)$. Hence all we have to show is that it is closed under addition and scalar multiplication.

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Theorem 5 ($\mathcal{L}^1(\mu)$ is a vector space) If $f, g \in \mathcal{L}^1(\mu)$ and $a, b \in \mathbb{C}$. Then

$$a \cdot f + b \cdot g \in \mathcal{L}^1(\mu) \quad \text{and} \quad \int_X (a \cdot f + b \cdot g)\, d\mu = a \cdot \int_X f\, d\mu + b \cdot \int_X g\, d\mu.$$

proof That $a \cdot f + b \cdot g$ is measurable follows from **Ch. 1.3 Corollary 17**. We now prove that $\int_X |a \cdot f + b \cdot g|\, d\mu \leq \infty$:
By **Ch.1.6 Proposition 4**, **Ch.1.6. Theorem 7** and the $\triangle \neq$ we have that

$$|a \cdot f + b \cdot g| \overset{\triangle\neq}{\leq} |a \cdot f| + |b \cdot g| \Rightarrow \int_X |a \cdot f + b \cdot g|\, d\mu \leq \int_X |a \cdot f| + |b \cdot g|\, d\mu \overset{\textbf{Ch.1.6.Th. 7}}{=}$$

$$\int_X |a \cdot f|\, d\mu + \int_X |b \cdot g|\, d\mu \overset{\textbf{Ch.1.6.Prop. 4c)}}{=} |a| \cdot \underbrace{\int_X |f|\, d\mu}_{<\infty} + |b| \cdot \underbrace{\int_X |g|\, d\mu}_{<\infty}.$$

Hence the integral is finite.
To prove the second equation we use the identity

$$f = (u^+ - u^-) + i \cdot (v^+ - v^-)$$

The idea is to reduce all possible cases to real valued positive functions. For this case the linearity of the integral follows from **Ch.1.6. Prop. 4 c)** and **Theorem 7**. A full proof can be found in *Rudin: Real and complex analysis, 2nd edition*, pg. 25,26. □

As for Riemann integrable functions we have:

Theorem 6 For $f \in \mathcal{L}^1(\mu)$ we have

$$|\int_X f\, d\mu| \leq \int_X |f|\, d\mu.$$

proof We know that $\mathbb{C} \ni \int_X f\, d\mu = r \cdot e^{i\theta} \Rightarrow |\int_X f\, d\mu| = r$. Setting $\alpha = e^{-i\theta}$ we then have

$$\begin{aligned} |\int_X f\, d\mu| = r \quad &= \quad \underbrace{(r \cdot e^{i\theta})}_{\int_X f\, d\mu} \cdot \underbrace{e^{-i\theta}}_{=\alpha} = \alpha \cdot \int_X f\, d\mu = \int_X \alpha \cdot f\, d\mu = \\ &\int_X \mathrm{Re}(\alpha \cdot f)\, d\mu \ + \ i \cdot \underbrace{\int_X \mathrm{Im}(\alpha \cdot f)\, d\mu}_{=0 \ \text{ as } \ |\int_X f\, d\mu| = r \in \mathbb{R}} \ = \\ &\int_X \mathrm{Re}(\alpha \cdot f)^+\, d\mu - \int_X \mathrm{Re}(\alpha \cdot f)^-\, d\mu \leq \\ &\int_X \mathrm{Re}(\alpha \cdot f)^+\, d\mu \leq \int_X |\alpha \cdot f|\, d\mu \overset{|\alpha|=1}{=} \int_X |f|\, d\mu. \quad \square \end{aligned}$$
