Math 103: Measure Theory and Complex Analysis
Fall 2018

11/12/18

Lecture 26

Chapter 5 - Local behavior

Last time: **Open Mapping Theorem:** Suppose Ω is a region and $f \in \mathcal{H}(\Omega)$. Then $f(\Omega)$ is either a point or a region.

Theorem 7 Suppose $f \in \mathcal{H}(\Omega)$ and $f'(z_0) \neq 0$ for some z_0 in Ω. Then there is an open neighborhood V of z_0 in Ω, such that

i) f is one-to-one on V.

ii) $W = f(V)$ is open

iii) f has a holomorphic inverse function f^{-1} on V.

proof As $f'(z_0) \neq 0$ we can choose $\epsilon > 0$ such that $D_\epsilon(z_0) \subset \Omega$ and such that

$$z \in D_\epsilon(z_0) \Rightarrow |f'(z) - f'(z_0)| \leq \frac{1}{2}|f'(z_0)|$$

Picture

This implies that $|f'(z)| \geq \frac{1}{2}|f'(z_0)|$ if $z \in D_\epsilon(z_0)$.
Let $V = D_\epsilon(z_0)$. Then $v \in V$ implies that $f'(z) \neq 0$. Also if $z_1, z_2 \in V$ then

$$f(z_2) - f(z_1) = \int_{[z_1,z_2]} f'(w)\,dw = \int_{[z_1,z_2]} f'(z_0)\,dw + \int_{[z_1,z_2]} f'(w) - f'(z_0)\,dw.$$

Using the statement above and the $\neq \triangle$ we can show that

$$|f(z_2) - f(z_1)| \geq \frac{1}{2}|f'(z_0)||z_2 - z_1| \neq 0. \qquad (*)$$

proof

This implies i). Furthermore ii) follows from the **Open Mapping Theorem**.
To prove iii), the existence of the holomorphic inverse let $g : W \to V$ be the inverse of f on V. We have to show that $g \in \mathcal{H}(W)$.
If $w_1 \neq w_2$ in W, then there is exactly one $z_i \in V$, such that $f(z_i) = w_i$ for $i \in \{1, 2\}$. Then

$$\frac{g(w_2) - g(w_1)}{w_2 - w_1} = \qquad\qquad .$$

Furthermore, if $w_2 \to w_1$ then $f(z_2) \to f(z_1)$ and by (*) $z_2 \to z_1$.
Since $f'(z_1) \neq 0$ we have

$$g'(w_1) =$$

Hence $g \in \mathcal{H}(W)$. □

Corollary 8 Suppose Ω is a region and $f \in \mathcal{H}(\Omega)$. If f is injective in Ω then $f'(z) \neq 0$ for all $z \in \Omega$ and f has a holomorphic inverse.

proof First, $f(\Omega)$ is a region. If $f'(z_0) = 0$ for some $z_0 \in \Omega$, since f is not constant we know

□

Example Let $\Omega = \{z \in \mathbb{C} : 0 < \text{Im}(z) < 2\pi\}$ and consider

$$f : \Omega \to \mathbb{C}, z = x + iy \mapsto f(z) = e^z = e^x \cdot (\cos(y) + i\sin(y)).$$

We checked that f is injective on Ω and $f(\Omega) = \mathbb{C} \setminus [0, \infty)$.
Then by **Corollary 8** we have that $f^{-1} \in \mathcal{H}(\mathbb{C} \setminus [0, \infty))$ and

$$f^{-1}(z) = \log(z) = \ln(|z|) + i \cdot \arg(z), \quad \text{where} \quad \arg(z) = \phi, \quad \text{such that} \quad z = |z| \cdot e^{i\phi}, \ \phi \in (0, 2\pi).$$

We verify $\log'(z) = \frac{1}{z}$.

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Chapter 6 - Cauchy's Theorem (adult version)

Preliminaries

A path γ defines a linear functional on the space of continuous functions $C(\gamma^\star)$:

$$\tilde{\gamma} : C(\gamma^\star) \to \mathbb{C}, f \mapsto \tilde{\gamma}(f) = \int_\gamma f(w)\,dw.$$

We recall that $\gamma_1 \simeq \gamma_2$, if $\tilde{\gamma}_1 = \tilde{\gamma}_2$.

If $\gamma_1, \gamma_2, \ldots, \gamma_n$ are paths in $\mathbb{C}$ then each $\tilde{\gamma}_i$ can be seen as a functional on $C(K)$ where

$$K = \bigcup_{i=1}^{n} \gamma_i^\star$$

Using a formal sum we can set $\Gamma = \gamma_1 \oplus \gamma_2 \oplus \ldots \oplus \gamma_n$ and define

$$\tilde{\Gamma}(f) = \tilde{\Gamma}(f) = \int_\Gamma f(w)\,dw := \sum_{i=1}^{n} \int_{\gamma_i} f(w)\,dw.$$

We say that two such sums Γ_1 and Γ_2 are equivalent if $\tilde{\Gamma}_1 = \tilde{\Gamma}_2$. We furthermore set $\Gamma^\star = \bigcup_{i=1}^{n} \gamma_i^\star$.

Picture

Definition 1 A **chain** in Ω is an equivalence class Γ of formal sums of paths in Ω. In particular $\Gamma^\star \subset \Omega$. We define

i) A **chain** Γ is called a **cycle** if Γ has a representation $\Gamma = \gamma_1 \oplus \gamma_2 \oplus \ldots \oplus \gamma_n$ where each γ_i is a closed path.

ii) If Γ is a cycle and $a \notin \Gamma^\star$, then $\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i} \cdot \int_\Gamma \frac{dz}{z-a}$.

iii) In a natural way we define $-\Gamma = (-\gamma_1) \oplus (-\gamma_2) \oplus \ldots \oplus (-\gamma_n)$ and $\Gamma_1 \oplus \Gamma_2$.

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Definition 2 A closed path $\gamma \in \Omega$ is said to be **homologous to zero** in Ω if for all $a \neq \Omega$ we have

$$\mathrm{Ind}_{\gamma}(a) = 0.$$

Examples

Note 3 If Ω is a convex region, then every closed path is homologous to zero. Hence $\mathrm{Ind}_{\Gamma}(a) = 0$ for all cycles $\Gamma \in \Omega$ and all $a \notin \Omega$.

proof

Cauchy's Theorem (adult version) (CTA) Let Ω be a domain and $f \in \mathcal{H}(\Omega)$. If Γ is a cycle in Ω such that $\mathrm{Ind}_{\Gamma}(a) = 0$ for all $a \notin \Omega$. Then

i) For all $z \in \Omega \backslash \Gamma^{\star}$ we have $\mathrm{Ind}_{\Gamma}(z) \cdot f(z) = \frac{1}{2\pi i} \cdot \int_{\Gamma} \frac{f(w)}{w - z}\, dw$.

ii) $\int_{\Gamma} f(z)\, dz = 0$.

iii) Furthermore if Γ_1 and Γ_2 are cycles such that $\mathrm{Ind}_{\Gamma_1}(a) = \mathrm{Ind}_{\Gamma_2}(a)$ for all $a \notin \Omega$, then

$$\int_{\Gamma_1} f(z)\, dz = \int_{\Gamma_2} f(z)\, dz.$$

proof see **Rudin: Real and complex analysis**, p. 218-220.

Cauchy's Integral Formula for an Annulus Suppose Ω is a domain containing

$$A = \{z \in \mathbb{C} : 0 \leq r < |z - a| < R < +\infty\}$$

and $f \in \mathcal{H}(\Omega)$. Let $\gamma_1(t) = a + r \cdot e^{it}$ and $\gamma_2(t) = a + R \cdot e^{it}$ for $t \in [0, 2 \cdot \pi]$. Then for all $z \in A$ we have

$$f(z) = \frac{1}{2\pi i} \cdot \int_{\gamma_2} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \cdot \int_{\gamma_1} \frac{f(w)}{w - z}\, dw \overset{\textbf{Def.}}{=} \frac{1}{2\pi i} \cdot \int_{\partial A} \frac{f(w)}{w - z}\, dw$$

Picture

proof Let $\Gamma = \gamma_2 - \gamma_1$. If $a \notin A \cup \Gamma^\star$, then

Now we can apply **CTA ii)** and obtain our result. □