Math 103: Measure Theory and Complex Analysis
Fall 2018

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Lecture 12

We proceed by constructing a premeasure on $\mathcal{A}$. To this end we first prove the following lemma.

Lemma 3 If $A, B \in \mathcal{R}$ and $A \times B = \biguplus_{i \in \mathbb{N}} A_i \times B_i$ where $A_i \times B_i \in \mathcal{R}$ then

$$\mu(A) \cdot \nu(B) = \sum_{i \in \mathbb{N}} \mu(A_i) \cdot \nu(B_i).$$

Picture

proof If $(x, y) \in X \times Y$ then

$\mathbb{1}_A(x) \cdot \mathbb{1}_B(y) =$

Now holding y fixed and integrating with respect to x we obtain

Then integrating with respect to y we obtain our statement. $\square$

We now show that there is a unique premeasure π on $\mathcal{A}$ that satisfies the formula of the lemma.

Lemma 4 There is a unique premeasure $\pi : \mathcal{A} \to [0, \infty]$ such that

$$\pi(A \times B) = \mu(A) \cdot \nu(B) \text{ for all } A \times B \in \mathcal{R}.$$

proof Clearly $\pi(\emptyset \times \emptyset) = 0$. It remains to show that π is well-defined and satisfies the second condition of a premeasure.

1.) π is well-defined

If $(A_i \times B_i)_{i=1}^n \subset \mathcal{R}$ and $(C_j \times D_j)_{j=1}^m \subset \mathcal{R}$ are families of mutually disjoint sets in $\mathcal{R}$, such that

$$\biguplus_{i=1}^{n} A_i \times B_i = A \times B = \biguplus_{j=1}^{m} C_j \times D_j.$$

Idea: Look at the refinement. Then

$A_i \times B_i =$ and $C_j \times D_j =$

Using the previous lemma twice we have that

$$\sum_{i=1}^{n} \mu(A_i) \cdot \nu(B_i) =$$

So the value of $\pi(A \times B)$ is independent of the partition of $A \times B$ into disjoint sets. This means that π is well-defined.

2.) π satisfies the second condition of a premeasure

We have to show: if $(R_i)_{i \in \mathbb{N}} \subset \mathcal{A}$ and $\boxed{R = \biguplus_{i \in \mathbb{N}} R_i \in \mathcal{A}}$ then $\pi(R) = \sum_{i \in \mathbb{N}} \pi(R_i)$.
Suppose that $R_i = A_i \times B_i \in \mathcal{A}$ and by the definition of $\mathcal{A}$, we have that

$$\biguplus_{i \in \mathbb{N}} A_i \times B_i = R = \biguplus_{j=1}^{m} C_j \times D_j \quad \text{where} \quad C_j \times D_j \in \mathcal{R}\,.$$

Note that it follows from the proof of **Lemma 2** that we may take a disjoint union of rectangles $C_j \times D_j \in \mathcal{R}$. Then like in the previous part we have

$$\pi(R) = \sum_{j=1}^{m} \mu(C_j) \cdot \nu(D_j) =$$

In total we have that π is a premeasure. $\square$

Definition 5 (product measure) The measure on $\mathcal{M} \otimes \mathcal{N} = \langle \mathcal{R} \rangle$ generated by π is called the **product of μ and ν** and is denoted by $\mu \times \nu$.

Remark 6 If μ and ν are σ finite, then so is π. In this case $\mu \times \nu$ is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ such that

$$\mu \times \nu(A \times B) = \mu(A) \cdot \nu(B) \quad \text{for all} \quad A \times B \in \mathcal{R}\,.$$

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Chapter 2.4. - Fubini-Tonelli Theorem

Outline Under certain simple conditions we can exchange the order of integration in the double integral. This is stated in the theorem of Fubini-Tonelli. To this end we first study the measurability of slice functions and prove the Monotone Class Lemma.

If $f : (X \times Y, \mathcal{M} \otimes \mathcal{N}) \to Z$ is a measurable function, then the slice fuctions which we obtain by fixing either the variable x or y are measurable with respect to $\mathcal{M}$ and $\mathcal{N}$, respectively.

Picture

We define the **slice functions**

$$f_x : Y \to Z, y \to f_x(y) = f(x, y) \quad \text{and} \quad f_y : X \to Z, x \to f_y(x) = f(x, y)$$

Similarly for sets $E \subset X \times Y$ we define the projection of the sections:

$$\text{Fixing } x: \ E_x = \{y \in Y \mid (x, y) \in E\}$$
$$\text{Fixing } y: \ E_y = \{x \in X \mid (x, y) \in E\}$$

This way we have

$$(\mathbb{1}_E)_x = \mathbb{1}_{E_x} \quad \text{and} \quad (\mathbb{1}_E)_y = \mathbb{1}_{E_y}.$$

Proposition 1 Suppose $E \in \mathcal{M} \otimes \mathcal{N}$ and $f : (X \times Y, \mathcal{M} \otimes \mathcal{N}) \to (Z, \mathcal{U})$ is a measurable function. Then

a) For all $(x, y) \in X \times Y$ we have $E_x \in \mathcal{N}$ and $E_y \in \mathcal{M}$.

b) f_x is $\mathcal{M}$ measurable and f_y is $\mathcal{N}$ measurable.

proof a) Let

$$\mathcal{Q} = \{E \subset X \times Y \mid E_x \in \mathcal{N} \text{ for all } x \in X \text{ and } E_y \in \mathcal{M} \text{ for all } y \in Y\}.$$

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Note that if $A \times B \in \mathcal{R}$, then

$$(A \times B)_x = \qquad\qquad\qquad\qquad \text{and} \quad (A \times B)_y =$$

Hence $\mathcal{R} \subset \mathcal{Q}$. We furthermore check that $\mathcal{Q}$ is a σ algebra:

a) Clearly $X \times Y \in \mathcal{R} \subset \mathcal{Q}$ and $\emptyset \in \mathcal{R} \subset \mathcal{Q}$

b) $A \in \mathcal{Q} \Rightarrow A^c \in \mathcal{Q}$: if $A_x \in \mathcal{N}$ and $A_y \in \mathcal{M}$ for all $(x, y) \in X \times Y$ then

$$A_x^c = \qquad\qquad\qquad\qquad \text{and } A_y^c =$$

c) $(A_i)_{i\in\mathbb{N}} \subset \mathcal{Q} \Rightarrow \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{Q}$: if $(A_i)_x \in \mathcal{N}$ and $(A_i)_y \in \mathcal{M}$ for all $(x, y) \in X \times Y$ then for all $(x, y) \in X \times Y$

$$\left(\bigcup_{i\in\mathbb{N}} A_i\right)_x = \qquad\qquad \text{and } \left(\bigcup_{i\in\mathbb{N}} A_i\right)_y = \qquad\qquad \Rightarrow \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{Q}\,.$$

b) To prove part b) we note that as $\mathcal{R} \subset \mathcal{Q}$, we know that $\langle\mathcal{R}\rangle = \mathcal{M} \otimes \mathcal{N} \subset \mathcal{Q}$. Hence if $Z \supset U \in \mathcal{U}$ and f is measurable, then for all $(x, y) \in X \times Y$

$$f^{-1}(U) \in \mathcal{M} \otimes \mathcal{N} \subset \mathcal{Q} \Rightarrow$$

Hence f_x and f_y are measurable by part a). □

Defintion 2 (monotone class) A subset $\mathcal{C} \subset \mathcal{P}(X)$ is called a **monotone class** if it is closed under increasing countable unions and decreasing countable sections.

Example Every σ algebra is a monotone class.

As the intersections of monotone classes are again monotone classes we define

Defintion 3 Let$\mathcal{E} \subset \mathcal{P}(X)$ be a subset of the power set of X then

$$\mathcal{C}(\mathcal{E}) := \bigcap_{\mathcal{C}' \text{ monotone class}, \mathcal{E} \subset \mathcal{C}'} \mathcal{C}'\,.$$

is called the **monotone class generated by** $\mathcal{E}$.