Math 103: Measure Theory and Complex Analysis
Fall 2018

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Lecture 11

Chapter 2.2 - Outer measures from premeasures

Outline We obtained the Lebesgue measure via the practical outer measure, which automatically induces a complete measure and a σ algebra. We can always construct outer measures from a simpler precursor of a measure, the **premeasure**.

Definition 1 (Algebra) Let X be a set. A collection of subsets $\mathcal{A} \subset \mathcal{P}(X)$ of X is called an **algebra** if

a) $X \in \mathcal{A}$

b) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ ($\mathcal{A}$ is closed under complements).

c) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$ ($\mathcal{A}$ is closed under **finite** unions).

Example $\mathcal{A} = \{\biguplus_{i=1}^n A_i \mid A_i = (a_i, b_i] \text{ or } A_i = (a_i, \infty), \quad \text{where } -\infty \le a_i \le b_i < +\infty\}$,

Set $\rho(\biguplus_{i=1}^n (a_i, b_i]) = \sum_{i=1}^n (b_i - a_i)$ then ρ^o is the Lebesgue measure (see **Def. 2**, **Prop. 3**).

Definition 2 (Premeasure) Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra. A **premeasure** on $\mathcal{A}$ is a map $\rho : \mathcal{A} \to [0, \infty]$, such that

a) $\rho(\emptyset) = 0$.

b) If $(A_i)_{i \in \mathbb{N}} \subset \mathcal{A}$ and $A = \biguplus_{i \in \mathbb{N}} A_i \in \mathcal{A}$ then $\rho(A) = \sum_{i \in \mathbb{N}} \rho(A_i)$.

Proposition 3 If $\rho : \mathcal{A} \to [0, \infty]$ is a premeasure on an algebra $\mathcal{A}$ of X, then

$$\rho^o(A) := \inf\{\sum_{i \in \mathbb{N}} \rho(A_i) \mid A \subset \bigcup_{i \in \mathbb{N}} A_i \text{ where } A_i \in \mathcal{A}\}$$

is an outer measure on X, such that

a) $\rho^o|_{\mathcal{A}} = \rho$.

b) $\mathcal{A} \subset \mathcal{M}^o$, the induced σ algebra of ρ^o measurable sets.

proof see *Folland, Real Analysis, 2nd edition*, Proposition 1.13.

Math 103: Measure Theory and Complex Analysis
Fall 2018

10/08/18

Theorem 4 (Extension of premeasures) Let $\rho : \mathcal{A} \to [0, \infty]$ is a premeasure on an algebra $\mathcal{A}$ of X. Let $\mathcal{M} = \langle \mathcal{A} \rangle$ be the σ algebra generated by $\mathcal{A}$. Then

a) there is a special measure $\mu : \mathcal{M} \to [0, \infty]$ such that $\mu|_{\mathcal{M}} = \rho$.

b) If $\nu : \mathcal{M} \to [0, \infty]$ such that $\nu|_{\mathcal{M}} = \rho$, then

$$\nu(A) \leq \mu(A) \text{ for all } A \in \mathcal{M} \quad \text{and} \quad \nu(A) = \mu(A) \text{ if } \mu(A) < \infty.$$

proof a) Clearly the special measure is $\boxed{\mu = \rho^o|_{\mathcal{M}} = \rho|_{\mathcal{A}}}$. This is possible as

$$\mathcal{A} \subset \mathcal{M}^o \Rightarrow$$

Hence we can obtain μ by restriction to $\mathcal{M}$.

b) Now let ν be another measure on $\mathcal{M}$, such that $\nu|_{\mathcal{A}} = \mu$. We first gather a few facts about ν and μ: if $E \subset \bigcup_{i \in \mathbb{N}} A_i$, where $A_i \in \mathcal{A}$, then

$$\nu(E) \overset{\nu \text{ measure}}{\leq}$$

By approximating $\mu(E)$ with sums of the form $\sum_{i \in \mathbb{N}} \mu(A_i)$ we obtain for all $E \in \mathcal{M}$:

$$\nu(E) \leq \mu(E). \tag{1}$$

Furthermore, if $\underline{A = \bigcup_{i \in \mathbb{N}} A_i}$, where $A_i \in \mathcal{A}$, then

$$\nu(A) \overset{\nu \text{ measure}}{=} \tag{2}$$

If $\underline{E \in \mathcal{M}}$ and $\underline{\mu(E) < \infty}$, then by the definition of ρ^o we can choose "approximating" $A_i \in \mathcal{A}$, such that $E \subset \bigcup_{i \in \mathbb{N}} A_i = A$ and

$$\mu(A) =$$

This implies that

$$\nu(E) \overset{(1)}{\leq}$$

Since $\epsilon > 0$ is arbitrary we obtain with (1) that $\nu(E) = \mu(E)$. $\square$

Math 103: Measure Theory and Complex Analysis
Fall 2018

10/08/18

Definition 5 (σ finite) If $\mathcal{A}$ is an algebra in X, then a premeasure $\rho : \mathcal{A} \to [0, \infty]$ is said σ **finite**, if

$$X = \bigcup_{i \in \mathbb{N}} A_i \quad \text{where} \quad A_i \in \mathcal{A} \quad \text{and} \quad \rho(A_i) < \infty \text{ for all } i \in \mathbb{N}\,.$$

Example $\mathbb{R} = \biguplus_{n \in \mathbb{Z}} (n, n+1]$.

Proposition (Uniqueness of extension) If $\rho : \mathcal{A} \to [0, \infty]$ is σ finite then it has a unique extension on $\langle \mathcal{A} \rangle = \mathcal{M}$.

proof Suppose $X = \bigcup_{i \in \mathbb{N}} A_i$. We already know that the statement is true for all sets of finite measure in $\mathcal{M}$. We also know that in $\mathcal{M}$ we can decompose $\bigcup_{i \in \mathbb{N}} A_i$ into a union of mutually disjoint measurable sets of finite measure (see proof of **Chapter 1.9. Theorem 5**). Therefore we may assume that $X = \biguplus_{i \in \mathbb{N}} A_i$, where the A_i satisfy the conditions of the previous definition. Then for all $E \in \mathcal{M}$ we have

$$\mu(E) =$$

Hence the extension is unique. $\square$

Question How is the Lebesgue measure a special case of this?

If we set

- $\mathcal{A} = \{\biguplus_{i=1}^{n} A_i \mid A_i = (a_i, b_i] \text{ or } A_i = (a_i, \infty), \quad \text{where } -\infty \leq a_i \leq b_i < +\infty\}$
- $\rho(\biguplus_{i=1}^{n} (a_i, b_i]) = \sum_{i=1}^{n} (b_i - a_i)$.

Then ρ is well-defined, i.e. does not depend on the decomposition into intervals. Furthermore the σ algebra generated by $\mathcal{A}$ is $\mathcal{B}(\mathbb{R})$ and its completion is Λ^o.
For details see *Folland, Real Analysis* Ch. 1.5.

Math 103: Measure Theory and Complex Analysis
Fall 2018

10/08/18

Chapter 2.3. Product measures

Outline Given two measure spaces $(X, \mathcal{M}, \mu)$ and $(X, \mathcal{N}, \nu)$ we can construct a natural measure $\mu \times \nu$ on $X \times Y$.

Definition 1 A **measurable rectangle** in $X \times Y$ is an element of the form

$$A \times B \quad \text{where} \quad A \in \mathcal{M}, B \in \mathcal{N} .$$

Let $\mathcal{R} = \{A \times B \mid A \in \mathcal{M}, B \in \mathcal{N}\}$ be the set of all measurable rectangles. We denote by $\mathcal{M} \otimes \mathcal{N} = \langle \mathcal{R} \rangle$ the σ algebra generated by the collection of measurable rectangles.

Picture

As a precursor we look at the algebra formed by finite unions of measurable rectangles.

Lemma 2 Let $\mathcal{A} = \{\bigcup_{i=1}^{n} A_i \mid A_i \in \mathcal{R}\}$. Then $\mathcal{A}$ is an algebra.

proof We check the conditions for an algebra.

a) $X \times Y \in \mathcal{A}$: this is clear.

b) $R \in \mathcal{A} \Rightarrow R^c \in \mathcal{A}$: consider two measurable rectangles $A \times B$ and $C \times D$. Then

   1.) $(A \times B) \cap (C \times D) =$ $\in \mathcal{R}$.
   This means that the intersection is again a rectangle.

   2.) $(A \times B)^c =$ $\mathcal{R}$.
   By 1.) this means that $(A \times B)^c$ can be written as the union of two rectangles.
   Picture

3.) $(A \times B) \cup (C \times D) =$.
This means that the union of two rectangles can be written as the disjoint union of three rectangles.

$$\text{and} \quad A \times B \cap C \times D = (A \cap C) \times (B \cap D) \in \mathcal{R}\,.$$

Picture

In general we have that for $R_i = A_i \times B_i \in \mathcal{R}$:

$$(\bigcup_{i=1}^{n} A_i \times B_i)^c = \bigcap_{i=1}^{n} (A_i \times B_i)^c \overset{2.)}{=} \bigcap_{i=1}^{n} (A_i^c \times Y) \cup (X \times B_i^c)$$

To prove that the last term can be written as a union of rectangles we note that a union of two rectangles can be written as the union of three disjoint rectangles by 3.). Hence taking the intersection is equal to taking the intersection of a disjoint union of rectangles, which gives again rectangles. In total the last term can be written as a disjoint union of rectangles.

c) $R_1, R_2 \in \mathcal{A} \Rightarrow R_1 \cup R_2 \in \mathcal{A}$: there is nothing to prove, as the union of two finite unions of rectangles is again a finite union of rectangles.

In total $\mathcal{A}$ is an algebra. □