Math 103: Measure Theory and Complex Analysis
Fall 2018

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Lecture 10

We conclude

Theorem 6 The Vitali set V is not Lebesgue measurable.

proof If $V \in \Lambda^o$ then $\lambda(V) = \lambda(V_i)$ for all i and by 2.)

$$1 = \lambda([0,1)) =$$

Hence $V \notin \Lambda^o$ as there can be no number $\lambda(V)$ that satisfies this equation. $\square$

Lemma 7 Let $A \subset V$, $A \in \Lambda^o$ be a measurable subset of the Vitali set. Then $\lambda(A) = 0$.

proof Set $A_i = A \oplus q_i \subset V \oplus q_i = V_i$. Then $\lambda(A) = \lambda(A_i)$ and

$$1 = \lambda([0,1)) \geq$$

This implies that $\lambda(A) = 0$, because otherwise we arrive at a contradiction $\square$.

Proposition 8 Let $A \subset \mathbb{R}$ with $\lambda^o(A) > 0$. Then A contains a non-measurable set.

proof Suppose $A \subset [0,1)$. Let $A_i = A \cap V_i$. If each $A_i \in \Lambda^o$, then $\lambda(A_i) = 0$ by the previous lemma. Therefore

$$0 =$$

In general we know that $\lambda^o(A \cap [n, n+1)) > 0$ for some $n \in \mathbb{Z}$ and

$$A \cap [n, n+1) - n \in [0,1).$$

Hence the general statement follows from the case where $A \subset [0,1)$. $\square$

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Cantor Set

Aim: We want to show that $\mathcal{B}(\mathbb{R}) \subsetneq \Lambda^o$. We first define the Cantor set C and then find a subset $A \subset C$ that is measurable, but not a Borel set.

Picture

Definition (Cantor set) Let $I_1^1 = (\frac{1}{3}, \frac{2}{3})$ and $I_1^2 = (\frac{1}{9}, \frac{2}{9}), I_2^2 = (\frac{7}{9}, \frac{8}{9})$. We continue, such that $I_1^n, I_2^n, \dots, I_{2^{n-1}}^n$ are the middle thirds of the 2^{n-1} intervals that make up

$$[0,1] \backslash \bigcup_{i=1}^{n-1} \bigcup_{j=1}^{2^{i-1}} I_j^i$$

The **Cantor set** is

$$C := [0,1] \backslash \bigcup_{n \geq 1} \bigcup_{j=1}^{2^{n-1}} I_j^n$$

Note We know that

- $\lambda(C) =$
- $C \neq \emptyset$ as the endpoints $\overline{I}_j^n \backslash I_j^n$ are all contained in C.
- C is compact, has no interior and no isolated points.
- C is uncountable.

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proof (C is uncountable) We can write any real number as an (infinite) decimal expansion. In the same way we can write every real number as an infinite ternary expansion using base three instead of base ten. If $a \in [0,1]$ then there is $(a_i)_{i\in\mathbb{N}}$, where $a_i \in \{0,1,2\}$, such that

$$a = \sum_{i\in\mathbb{N}} \frac{a_i}{3^i}$$

Furthermore this expression is unique unless $a = \frac{z}{3^n}$, where $z \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Example $\frac{1}{3} \simeq (1,0,0,0,\ldots) \simeq (0,2,2,2,\ldots)$.

We will now use the ternary expansion to show that C is uncountable: To this end set

$$N = \begin{cases} \infty & \text{if} \quad a_i \neq 1 \text{ for all } i \in \mathbb{N} \\ j & \phantom{\text{if}} \quad a_j = 1 \;\text{ and }\; a_i \neq 1 \text{ for all } i < j. \end{cases}$$

Now define $\varphi : [0,1] \to [0,1]$,

$$\varphi(a) := \sum_{i=1}^{N} \frac{b_i}{2^i} \quad \text{where} \quad b_i := \begin{cases} 0 & & a_i = 0 \\ 1 & \text{if} & a_i = 2 \\ 1 & & a_i = a_N \end{cases}.$$

The function φ is callled a **Cantor function** or also a **Devil's staircase**. It is illustrated below.

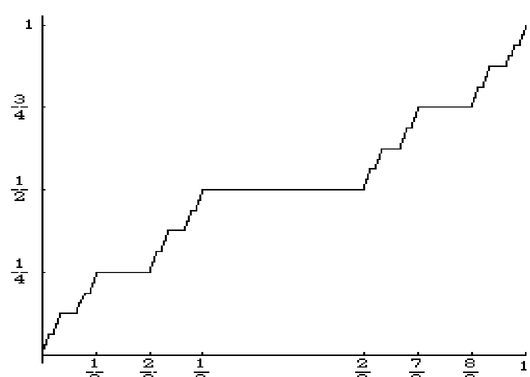

Figure 1: The Cantor function φ.

Note that φ is well-defined: $\varphi(a)$ does not depend on the representation of a in terms of its ternary expansion. It is monotone and surjective, hence continuous.

If a is the left-hand endpoint of $\overline{I}_j^n$ then

$$\varphi(x) = \varphi(a) \quad \text{for all} \quad x \in \overline{I}_j^n \tag{1}$$

Hence $\varphi(C) = [0,1]$ and in particular, C is uncountable. □

We now define $f : [0,1] \to [0,2]$ by

$$f(x) := x + \varphi(x)$$

Claim: f is a homeomorphism.

proof f is continuous and surjective. Furthermore $[0,1]$ is compact and $[0,2]$ is Hausdorff. We recall the following theorem from topology:

Theorem Let $f : X \to Y$ be a bijective continuous function. If X is compact, and Y is Hausdorff, then $f : X \to Y$ is a homeomorphism.

It is therefore sufficient to verify that f is injective. Let without loss of generality $y > x$. Then

Hence $f(y) > f(x)$. By the same arguments for $x > y$ we have that f is injective. □