Math 73/103: Measure Theory and Complex Analysis
Fall 2018 - Homework 5

1. Let Ω be a domain. Show that $f_n \to f$ uniformly on compact subsets of Ω if and only if $f_n \to f$ uniformly on every closed disk contained in Ω.

2. Prove the Dog Walking Lemma: let γ_0 and γ_1 be closed paths. Let $a \in \mathbf{C}$ and suppose that

$$|\gamma_1(t) - \gamma_0(t)| < |a - \gamma_0(t)| \quad \text{for } t \in [0,1].$$

Conclude that $\operatorname{Ind}_{\gamma_0}(a) = \operatorname{Ind}_{\gamma_1}(a)$. In other words, γ_0 and γ_1 wrap around a exactly the same number of times. (So if someone walking their dog in a park with a lamp post in the center never comes nearer the lamp post than the length of the leash, they both circle the lamp post the same number of times.)

Hint: Note that $a \notin \gamma_k$ for $k = 0, 1$ and let $\gamma(t) = \frac{\gamma_1(t)-a}{\gamma_0(t)-a}$. Observe that $\gamma^* \subset D = B_1(1)$, and conclude that $\operatorname{Ind}_\gamma(0) = 0$.

3. **Rudin on pages 227–230:** 2, 3, 4, 5, 13 and 20.

- For Problem 2: The Baire Category Theorem implies that if $\mathbb{C} = \bigcup F_n$ when each F_n closed, then some F_n has interior.
- For Problem 4: Estimate $|f^{k+1}(z)|$.
- For Problem 5: The hypotheses of the problem don't allow us to conclude even that the limit function $f = \lim_n f_n$ is continuous. Instead, you'll have to prove that $\{ f_n \}$ is uniformly Cauchy. You may want to use (and prove) that if $|g_n(z)| \leq M$ for all $z \in \gamma^*$ and g_n converges pointwise to a 0, then
$$\int_\gamma g(z)\, dz \to 0.$$
- For Problem 20: Notice that $f_n' \to f'$ uniformly on compact subsets of Ω. Show this implies $f_n'/f_n \to f'/f$ uniformly on any γ^* provided $f \neq 0$ on γ^*.

4. Prove Rouché's Theorem: suppose that f and g are analytic in a simply connected domain containing a simple closed contour Γ, and that for $z \in \Gamma^*$,

$$|f(z) - g(z)| < |f(z)|.$$

Notice that this implies neither f nor g has zeros on Γ. Show that $N_f = N_g$, where N_f is the number of zeros of f inside Γ counted up to multiplicity. You may assume $Ind_\Gamma(a) = 1$ for all a inside of Γ. (Use the Dog Walking Lemma and the observation $N_f = \operatorname{Ind}_{f(\Gamma)}(0)$.)