**Math 73/103: Measure Theory and Complex Analysis**
**Fall 2018 - Homework 2**

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1. Page 32 of Rudin, problem #6. (Note that we have already shown that $\mathcal{M}$ is a $\sigma$-algebra so there is no need to show it again.)

2. Page 32 of Rudin, problem #7.

3. Page 32 of Rudin, problem #10.

4. Page 32 of Rudin, problem #12. (This is easy if $f$ is bounded.)

5. Suppose that $Y$ is a topological space and that $\mathcal{M}$ is a $\sigma$-algebra in $Y$ containing all the Borel sets. Suppose in addition, $\mu$ is a measure on $(Y, \mathcal{M})$ such that for all $E \in \mathcal{M}$ we have

$$\mu(E) = \inf\{\, \mu(V) : V \text{ is open and } E \subset V \,\}. \tag{1}$$

Suppose also that

$$Y = \bigcup_{n=1}^{\infty} Y_n \quad \text{with } \mu(Y_n) < \infty \text{ for all } n \geq 1. \tag{2}$$

One says that $\mu$ is a *$\sigma$-finite outer regular measure* on $(Y, \mathcal{M})$.

(a) Show that Lebesgue measure $m$ is a $\sigma$-finite outer regular measure on $(\mathbb{R}, \mathcal{M})$.

(b) Suppose $E$ is a $\mu$-measurable subset of $Y$.

  (i) Given $\varepsilon > 0$, show that there is an open set $V \subset Y$ and a closed set $F \subset Y$ such that $F \subset E \subset V$ and $\mu(V \setminus F) < \varepsilon$.

  (ii) Show that there is a $G_\delta$-set $G \subset Y$ and a $F_\sigma$-set $A \subset Y$ such that $A \subset E \subset G$ and $\mu(G \setminus A) = 0$.

(c) Argue that $(\mathbb{R}, \mathcal{M}, m)$ is the completion of the restriction of Lebesgue measure to the Borel sets in $\mathbb{R}$.

6. Let $m$ be Lebesgue measure on $\mathbb{R}$ and suppose that $E$ is a set of finite measure. Given $\varepsilon > 0$, show that there is a finite *disjoint* union $F$ of open intervals such that $m(E \triangle F) < \varepsilon$ where $E \triangle F := (E \setminus F) \cup (F \setminus E)$ is the symmetric difference. (This illustrates the first of Littlewood's three principles: "Every Lebesgue measurable set is nearly a disjoint union of open intervals".)

7. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $(X, \mathcal{M}_0, \mu_0)$ be its completion.

(a) Let $f : X \to \mathbb{C}$ be a $\mu_0$-measurable function and assume that $g : X \to \mathbb{C}$ is a $\mu$-measurable function such that $f = g$ a.e. $[\mu_0]$. Is there necessarily a $\mu$-null set $N$ such that $f(x) = g(x)$ for all $x \notin N$?

(b) If $f : X \to \mathbb{C}$ is $\mu_0$-measurable, show that there is a $\mu$-measurable function $g : X \to \mathbb{C}$ such that $f = g$ a.e. $[\mu_0]$.

(c) What does this result say about Lebesgue measurable functions and Borel functions on $\mathbb{R}$? (Compare with problem #14 on page 59 of Rudin.)