Course 221: Hilary Term 2007
Section 6: The Extended Real Number System

David R. Wilkins

Contents

6 The Extended Real Number System **2**

6.1 The Extended Real Line 2

6.2 Summation over Countable Sets 4

6.3 Summable Functions 10

6 The Extended Real Number System

6.1 The Extended Real Line

It is often convenient to make use of the *extended real line* $[-\infty, +\infty]$. This is the set $\mathbb{R} \cup \{-\infty, +\infty\}$ obtained on adjoining to the real line $\mathbb{R}$ two extra elements $+\infty$ and $-\infty$ that represent points at 'positive infinity' and 'negative infinity' respectively. We define

$$c + (+\infty) = (+\infty) + c = +\infty$$

and

$$c + (-\infty) = (-\infty) + c = -\infty$$

for all real numbers c. We also define products of non-zero real numbers with these extra elements $\pm\infty$ so that

$$\begin{aligned} c \times (+\infty) &= (+\infty) \times c = +\infty \quad \text{when } c > 0, \\ c \times (-\infty) &= (-\infty) \times c = -\infty \quad \text{when } c > 0, \\ c \times (+\infty) &= (+\infty) \times c = -\infty \quad \text{when } c < 0, \\ c \times (-\infty) &= (-\infty) \times c = +\infty \quad \text{when } c < 0, \end{aligned}$$

We also define

$$0 \times (+\infty) = (+\infty) \times 0 = 0 \times (-\infty) = (-\infty) \times 0 = 0,$$

and

$$\begin{aligned} (+\infty) \times (+\infty) = (-\infty) \times (-\infty) = +\infty, \\ (+\infty) \times (-\infty) = (-\infty) \times (+\infty) = -\infty. \end{aligned}$$

The sum of $+\infty$ and $-\infty$ is not defined. We define $-(+\infty) = -\infty$ and $-(-\infty) = +\infty$). The difference $p - q$ of two extended real numbers is then defined by the formula $p - q = p + (-q)$, unless $p = q = +\infty$ or $p = q = -\infty$, in which cases the difference of the extended real numbers p and q is not defined.

We extend the definition of inequalities to the extended real line in the obvious fashion, so that $c < +\infty$ and $c > -\infty$ for all real numbers c, and $-\infty < +\infty$.

Given any real number c, we define

$$\begin{aligned} [c, +\infty] &= [c, +\infty) \cup \{+\infty\} = \{p \in [-\infty, \infty] : p \geq c\}, \\ (c, +\infty] &= (c, +\infty) \cup \{+\infty\} = \{p \in [-\infty, \infty] : p > c\}, \\ [-\infty, c] &= (-\infty, c] \cup \{-\infty\} = \{p \in [-\infty, \infty] : p \leq c\}, \\ [-\infty, c) &= (-\infty, c) \cup \{-\infty\} = \{p \in [-\infty, \infty] : p < c\}. \end{aligned}$$

There is an order-preserving bijective function $\varphi\colon [-\infty, +\infty] \to [-1, 1]$ from the extended real line $[-\infty, +\infty]$ to the closed interval $[-1, 1]$ which is defined such that $\varphi(+\infty) = 1$, $\varphi(-\infty) = -1$, and $\varphi(c) = \dfrac{c}{1 + |c|}$ for all real numbers c. Let us define $\rho(p, q) = |\varphi(q) - \varphi(p)|$ for all extended real numbers p and q. Then the set $[-\infty, +\infty]$ becomes a metric space with distance function ρ. Moreover the function $\varphi\colon [-\infty, +\infty] \to [-1, 1]$ is a homeomorphism from this metric space to the closed interval $[-1, 1]$. It follows directly from this that $[-\infty, +\infty]$ is a compact metric space. Moreover an infinite sequence $(p_j : j \in \mathbb{N})$ of extended real numbers is convergent if and only if the corresponding sequence $(\varphi(p_j) : j \in \mathbb{N})$ of real numbers is convergent.

Given any non-empty set S of extended real numbers, we can define $\sup S$ to be the least extended real number p with the property that $s \leq p$ for all $s \in S$. If the set S does not contain the extended real number $+\infty$, and if there exists some real number B such that $s \leq B$ for all $s \in S$, then $\sup S < +\infty$; otherwise $\sup S = +\infty$. Similarly we define $\inf S$ to be the greatest extended real number p with the property that $s \geq p$ for all $s \in S$. If the set S does not contain the extended real number $-\infty$, and if there exists some real number A such that $s \geq A$ for all $s \in S$, then $\inf S > +\infty$; otherwise $\inf S = -\infty$. Moreover

$$\varphi(\sup S) = \sup \varphi(S) \text{ and } \varphi(\inf S) = \inf \varphi(S),$$

where $\varphi\colon [-\infty, +\infty] \to [-1, 1]$ is the homeomorphism defined such that $\varphi(+\infty) = 1$, $\varphi(-\infty) = -1$ and $\varphi(c) = c(1 + |c|)^{-1}$ for all real numbers c.

Given any sequence $(p_j : j \in \mathbb{N})$ of extended real numbers, we define the *upper limit* $\limsup\limits_{j\to+\infty} p_j$ and the *lower limit* $\limsup\limits_{j\to+\infty} p_j$ of the sequence so that

$$\limsup_{j\to+\infty} p_j = \lim_{j\to+\infty} \sup\{p_k : k \geq j\}, \quad \liminf_{j\to+\infty} p_j = \lim_{j\to+\infty} \inf\{p_k : k \geq j\}.$$

Every sequence of extended real numbers has both an upper limit and a lower limit. Moreover an infinite sequence of extended real numbers converges to some extended real number if and only if the upper and lower limits of the sequence are equal. (These results follow easily from the corresponding results for bounded sequences of real numbers, on using the identities

$$\varphi(\limsup_{j\to+\infty} p_j) = \limsup_{j\to+\infty} \varphi(p_j), \quad \varphi(\liminf_{j\to+\infty} p_j) = \liminf_{j\to+\infty} \varphi(p_j),$$

where $\varphi\colon [-\infty, +\infty] \to [-1, 1]$ is the homeomorphism defined above.)

The function that sends a pair (p, q) of extended real numbers to the extended real number $p + q$ is not defined when $p = +\infty$ and $q = -\infty$, or when $p = -\infty$ and $q = +\infty$ but is continuous elsewhere. The function that sends a pair (p, q) of extended real numbers to the extended real number pq is defined everywhere. This function is discontinuous when $p = \pm\infty$ and $q = 0$, and when $p = 0$ and $q = \pm\infty$. It is continuous for all other values of the extended real numbers p and q.

Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of extended real numbers which does not include both the values $+\infty$ and $-\infty$, and let $p_k = \sum_{j=0}^{k} a_j$ for all natural numbers k. If the infinite sequence $p_1, p_2, p_3, \ldots$ of extended real numbers converges in the extended real line $[-\infty, +\infty]$ to some extended real number p, then this value p is said to be the *sum* of the infinite series $\sum\limits_{j=1}^{+\infty} a_j$, and we write $\sum\limits_{j=1}^{+\infty} a_j = p$.

It follows easily from this definition that if $+\infty$ is one of the values of the infinite series $a_1, a_2, a_3, \ldots$, then $\sum\limits_{j=1}^{+\infty} a_j = +\infty$. Similarly if $-\infty$ is one of the values of this infinite series then then $\sum\limits_{j=1}^{+\infty} a_j = -\infty$. Suppose that the members of the sequence $a_1, a_2, a_3, \ldots$ are all real numbers. Then $\sum\limits_{j=1}^{+\infty} a_n = +\infty$ if and only if, given any real number B, there exists some real number N such that $\sum\limits_{j=1}^{k} a_n > B$ whenever $k \geq N$. Similarly $\sum\limits_{j=1}^{+\infty} a_j = -\infty$ if and only if, given any real number A, there exists some real number N such that $\sum\limits_{j=1}^{k} a_j < A$ whenever $k \geq N$.

6.2 Summation over Countable Sets

Let S be a countable set, and let $\lambda \colon S \to [0, +\infty]$ be a function on the set S that takes values in the set $[0, +\infty]$ of non-negative extended real numbers. We define

$$\sum_{s \in S} \lambda(s) = \sup \left\{ \sum_{s \in F} \lambda(s) : F \text{ is finite and } F \subset S \right\}.$$

We define $\sum_{s \in S} \lambda(s) = 0$ when $S = \emptyset$.

If the set S is finite and non-empty, then there exist distinct elements $s_1, s_2, \ldots, s_m$ of S such that $S = \{s_1, s_2, \ldots, s_m\}$. Then

$$\sum_{s \in S} \lambda(s) = \lambda(s_1) + \lambda(s_2) + \cdots + \lambda(s_m).$$

Proposition 6.1 *Let S be a countable infinite set, let $\lambda\colon S \to [0, +\infty]$ be a function on S with values in the set $[0, +\infty]$ of non-negative extended real numbers, and let $\varphi\colon \mathbb{N} \to S$ be a bijective function mapping the set $\mathbb{N}$ of natural numbers onto S. Then*

$$\sum_{s\in S} \lambda(s) = \sum_{j=1}^{+\infty} \lambda(\varphi(j)).$$

Thus if $s_1, s_2, s_3, \ldots$ be an infinite sequence of distinct elements of S that includes every element of S, then

$$\sum_{s\in S} \lambda(s) = \sum_{j=1}^{+\infty} \lambda(s_j).$$

Proof Given any finite subset F of S, there exists some natural number N such that $F \subset \{\varphi(j) : j \in \mathbb{N} \text{ and } j \le N\}$. Then

$$\sum_{s\in F} \lambda(s) \le \sum_{j=1}^{N} \lambda(\varphi(j)) \le \sum_{j=1}^{+\infty} \lambda(\varphi(j)).$$

It follows that

$$\sum_{s\in S} \lambda(s) = \sup \left\{ \sum_{s\in F} \lambda(s) : F \text{ is finite and } F \subset S \right\} \le \sum_{j=1}^{+\infty} \lambda(\varphi(j)).$$

But

$$\sum_{j=1}^{N} \lambda(\varphi(j)) \le \sum_{s\in S} \lambda(s)$$

for all natural numbers N, and therefore

$$\sum_{j=1}^{+\infty} \lambda(\varphi(j)) = \lim_{N\to+\infty} \sum_{j=1}^{N} \lambda(\varphi(j)) \le \sum_{s\in S} \lambda(s).$$

It follows that $\sum\limits_{s\in S} \lambda(s) = \sum\limits_{j=1}^{+\infty} \lambda(\varphi(j))$, as required. ∎

Corollary 6.2 *Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of non-negative real numbers, and let $\varphi\colon \mathbb{N} \to \mathbb{N}$ be a permutation of the set of natural numbers. Then $\sum\limits_{j=1}^{+\infty} a_j = \sum\limits_{j=1}^{+\infty} a_{\varphi(j)}$. Thus the sum of any infinite series of non-negative real numbers has a value which is independent of the order of summation.*

Proof Let $\lambda\colon \mathbb{N} \to \mathbb{R}$ be the function defined such that $\lambda(j) = a_j$ for all natural numbers j. It follows immediately from Proposition 6.1 that

$$\sum_{j=1}^{+\infty} a_{\varphi(j)} = \sum_{j=1}^{+\infty} \lambda(\varphi(j)) = \sum_{j\in\mathbb{N}} \lambda(j).$$

Thus the sum of the infinite series $\sum\limits_{j=1}^{+\infty} a_{\varphi(j)}$ has a value that is the same for all permutations $\varphi\colon \mathbb{N} \to \mathbb{N}$ of the set $\mathbb{N}$ of natural numbers, and is therefore equal to $\sum\limits_{j=1}^{+\infty} a_j$. ∎

Let $\mathcal{C}$ be a collection of sets. We say that the sets in this collection are *pairwise disjoint* if and only if the intersection of any two distinct sets in the collection is the empty set.

Proposition 6.3 *Let $\mathcal{C}$ be a countable collection of countable sets, let V be the union $\bigcup_{S\in\mathcal{C}} S$ of the sets belonging to the collection $\mathcal{C}$, and let $\lambda\colon V \to [0,+\infty]$ be a function on V taking values in the set $[0,+\infty]$ of non-negative extended real numbers. Suppose that the sets in the collection $\mathcal{C}$ are pairwise disjoint. Then*

$$\sum_{s\in V} \lambda(s) = \sum_{S\in\mathcal{C}} \left(\sum_{s\in S} \lambda(s) \right).$$

Proof The result follows immediately if $\mathcal{C} = \emptyset$. It suffices therefore to prove the result when the collection $\mathcal{C}$ is non-empty.

First we prove that

$$\sum_{s\in V} \lambda(s) \leq \sum_{S\in\mathcal{C}} \left(\sum_{s\in S} \lambda(s) \right).$$

Let F be a finite subset of V. Then the number of sets in the collection $\mathcal{C}$ which have non-empty intersection with F is finite. (Indeed the number of such sets cannot exceed the number of elements in the finite set F.) Let the number of such sets be m, and let these sets be $S_1, S_2, \ldots, S_m$. Then the sets $S_1, S_2, \ldots, S_m$ are pairwise disjoint (so that $S_i \cap S_j = \emptyset$ whenever $i \neq j$), and $F = \bigcup\limits_{j=1}^{m} (F \cap S_j)$, and therefore

$$\sum_{s\in F} \lambda(s) = \sum_{j=1}^{m} \left(\sum_{s\in F\cap S_j} \lambda(s) \right) \leq \sum_{j=1}^{m} \left(\sum_{s\in S_j} \lambda(s) \right) \leq \sum_{S\in\mathcal{C}} \left(\sum_{s\in S} \lambda(s) \right).$$

It follows from this that

$$\sum_{s\in V}\lambda(s) = \sup\left\{\sum_{s\in F}\lambda(s) : F \text{ is finite and } F\subset V\right\} \leq \sum_{S\in\mathcal{C}}\left(\sum_{s\in S}\lambda(s)\right).$$

We now prove the reverse inequality

$$\sum_{S\in\mathcal{C}}\left(\sum_{s\in S}\lambda(s)\right) \leq \sum_{s\in V}\lambda(s).$$

If $\sum_{s\in V}\lambda(s) = +\infty$ there is nothing to prove. Suppose that $\sum_{s\in V}\lambda(s) < +\infty$. Then $\sum_{s\in S}\lambda(s) \leq \sum_{s\in V}\lambda(s) < +\infty$ for all $S\in\mathcal{C}$. Let $S_1, S_2, \ldots, S_r$ be distinct sets belonging to the collection $\mathcal{C}$, and let ε be some positive real number. Then, for each integer j between 1 and r, there exists a finite subset F_j of S_j such that

$$\sum_{s\in S_j}\lambda(s) - \frac{\varepsilon}{r} < \sum_{s\in F_j}\lambda(s).$$

(Indeed $\sum_{s\in S_j}\lambda(s)$ is by definition the least upper bound of the sums of the values of the function λ taken over finite subsets of S_j, and may therefore be approximated to within an error of ε/r by the sum taken over some sufficiently large finite subset of S_j.) Let $F = F_1\cup F_2\cup\ldots\cup F_r$. Now the sets $S_1, S_2, \ldots, S_r$ are pairwise disjoint, and therefore so are the sets $F_1, F_2, \ldots, F_r$. It follows that

$$\sum_{j=1}^{r}\left(\sum_{s\in S_j}\lambda(s)\right) - \varepsilon < \sum_{j=1}^{r}\left(\sum_{s\in F_j}\lambda(s)\right) = \sum_{s\in F}\lambda(s) \leq \sum_{s\in V}\lambda(s).$$

The inequality

$$\sum_{j=1}^{r}\left(\sum_{s\in S_j}\lambda(s)\right) - \varepsilon < \sum_{s\in V}\lambda(s).$$

therefore holds for any positive value of ε, no matter how small. It follows that

$$\sum_{j=1}^{r}\left(\sum_{s\in S_j}\lambda(s)\right) \leq \sum_{s\in V}\lambda(s).$$

Thus

$$\sum_{S\in\mathcal{F}}\left(\sum_{s\in S}\lambda(s)\right) \leq \sum_{s\in V}\lambda(s)$$

for any finite collection $\mathcal{F}$ of sets whose members belong to $\mathcal{C}$, and therefore

$$\begin{aligned}\sum_{S\in\mathcal{C}}\left(\sum_{s\in S}\lambda(s)\right) &= \sup\left\{\sum_{S\in\mathcal{F}}\left(\sum_{s\in S}\lambda(s)\right) : \mathcal{F} \text{ is finite and } \mathcal{F}\subset\mathcal{C}\right\}\\ &\leq \sum_{s\in V}\lambda(s).\end{aligned}$$

But we have already shown that $\displaystyle\sum_{s\in V}\lambda(s) \leq \sum_{S\in\mathcal{C}}\left(\sum_{s\in S}\lambda(s)\right)$. Therefore

$$\sum_{s\in V}\lambda(s) = \sum_{S\in\mathcal{C}}\left(\sum_{s\in S}\lambda(s)\right),$$

as required. ∎

Corollary 6.4 *Let S and T be countable sets, and let $\lambda\colon S\times T\to[0,+\infty]$ be a function on $S\times T$ taking values in the set $[0,+\infty]$ of non-negative extended real numbers. Then*

$$\sum_{s\in S}\left(\sum_{t\in T}\lambda(s,t)\right) = \sum_{(s,t)\in S\times T}\lambda(s,t) = \sum_{t\in T}\left(\sum_{s\in S}\lambda(s,t)\right).$$

Proof For each $s\in S$, let $T_s = \{(s,t) : t\in T$. Then the Cartesian product $S\times T$ is the disjoint union of the sets T_s as s ranges over the elements of the countable set S. It follows from Proposition 6.3 that

$$\sum_{s\in S}\left(\sum_{t\in T}\lambda(s,t)\right) = \sum_{s\in S}\left(\sum_{(s,t)\in T_s}\lambda(s,t)\right) = \left(\sum_{(s,t)\in S\times T}\lambda(s,t)\right).$$

Similarly

$$\sum_{t\in T}\left(\sum_{s\in S}\lambda(s,t)\right) = \left(\sum_{(s,t)\in S\times T}\lambda(s,t)\right),$$

as required. ∎

Lemma 6.5 *Let S be a countable set, and let $\lambda\colon S\to[0,+\infty]$ and $\mu\colon S\to[0,+\infty]$ be functions on S taking values in the set $[0,+\infty]$ of non-negative extended real numbers. Then*

$$\sum_{s\in S}(\lambda(s)+\mu(s)) = \sum_{s\in S}\lambda(s) + \sum_{s\in S}\mu(s).$$

Proof Let F be a finite subset of S. Then

$$\sum_{s\in F}(\lambda(s)+\mu(s)) = \sum_{s\in F}\lambda(s) + \sum_{s\in F}\mu(s) \leq \sum_{s\in S}\lambda(s) + \sum_{s\in S}\mu(s).$$

Now $\sum_{s\in S}(\lambda(s)+\mu(s))$ is by definition the supremum of the quantities

$$\sum_{s\in F}(\lambda(s)+\mu(s))$$

as F ranges over all finite subsets of S. It follows that

$$\sum_{s\in S}(\lambda(s)+\mu(s)) \leq \sum_{s\in S}\lambda(s) + \sum_{s\in S}\mu(s).$$

It remains therefore to show that the reverse inequality is also valid.

Suppose that $\sum_{s\in S}(\lambda(s)+\mu(s)) < +\infty$. Then $\sum_{s\in S}\lambda(s) < +\infty$ and $\sum_{s\in S}\mu(s) < +\infty$. Let ε be some given positive real number. Then there exist finite subsets F_1 and F_2 of S such that

$$\sum_{s\in F_1}\lambda(s) > \sum_{s\in S}\lambda(s) - \tfrac{1}{2}\varepsilon \quad \text{and} \quad \sum_{s\in F_2}\mu(s) > \sum_{s\in S}\mu(s) - \tfrac{1}{2}\varepsilon.$$

Let $F = F_1 \cup F_2$. Then

$$\begin{aligned}\sum_{s\in S}(\lambda(s)+\mu(s)) &\geq \sum_{s\in F}(\lambda(s)+\mu(s)) = \sum_{s\in F}\lambda(s) + \sum_{s\in F}\mu(s)\\ &\geq \sum_{s\in F_1}\lambda(s) + \sum_{s\in F_2}\mu(s) > \sum_{s\in S}\lambda(s) + \sum_{s\in S}\mu(s) - \varepsilon.\end{aligned}$$

The inequality

$$\sum_{s\in S}(\lambda(s)+\mu(s)) > \sum_{s\in S}\lambda(s) + \sum_{s\in S}\mu(s) - \varepsilon$$

therefore holds, irrespective of the value of the positive real quantity ε. It follows that

$$\sum_{s\in S}(\lambda(s)+\mu(s)) \geq \sum_{s\in S}\lambda(s) + \sum_{s\in S}\mu(s),$$

when $\sum_{s\in S}(\lambda(s)+\mu(s)) < +\infty$. This inequality obviously holds also when the sum on the left hand side has the value $+\infty$. Therefore

$$\sum_{s\in S}(\lambda(s)+\mu(s)) = \sum_{s\in S}\lambda(s) + \sum_{s\in S}\mu(s),$$

as required. ∎

6.3 Summable Functions

Definition Let $f\colon S \to \mathbb{C}$ be a function defined on a countable set S and taking values in the field $\mathbb{C}$ of complex numbers. The function f is said to be *summable* if the set S, provided that

$$\sum_{s\in S} |f(s)| < +\infty.$$

We shall show that we can attach a well-defined value to the sum

$$\sum_{s\in S} f(s)$$

of the values of a summable function defined on a countable set S. This result is equivalent to the well-known theorem of analysis which states that an infinite series of real or complex numbers has a sum which is independent of the order of summation, provided that the infinite series is absolutely convergent.

Let S be a countable set, and let $f\colon S \to \mathbb{C}$ be a function on S with values in the field of complex numbers. We can write

$$f(s) = x^+(s) - x^-(s) + iy^+(s) - iy^-(s),$$

where

$$x^+(s) = \max(\mathrm{Re}[f(s)], 0), \quad x^-(s) = \max(-\mathrm{Re}[f(s)], 0),$$

$$y^+(s) = \max(\mathrm{Im}[f(s)], 0), \quad y^-(s) = \max(-\mathrm{Im}[f(s)], 0).$$

Then x^+, x^-, y^+ and y^- are functions on S with values in the set of non-negative real numbers. We can therefore sum these functions over the set S to obtain well-defined extended real numbers

$$\sum_{s\in S} x^+(s), \quad \sum_{s\in S} x^-(s), \quad \sum_{s\in S} y^+(s), \quad \text{and} \quad \sum_{s\in S} y^-(s)$$

(which may be finite or infinite). Now

$$x^+(s) \le |f(s)|, \quad x^-(s) \le |f(s)|, \quad y^+(s) \le |f(s)|, \quad \text{and} \quad y^-(s) \le |f(s)|$$

and

$$|f(s)| \le x^+(s) + x^-(s) + y^+(s) + y^-(s)$$

for all $s \in S$. It follows that $\sum_{s\in S} |f(s)| < +\infty$ if and only if

$$\sum_{s\in S} x^+(s) < +\infty, \quad \sum_{s\in S} x^-(s) < +\infty,$$

$$\sum_{s \in S} y^+(s) < +\infty, \quad \text{and} \quad \sum_{s \in S} y^-(s) < +\infty.$$

Definition Let S be a countable set, and let $f\colon S \to \mathbb{C}$ be a summable function on S. We define

$$\sum_{s \in S} f(s) = \sum_{s \in S} x^+(s) - \sum_{s \in S} x^-(s) + i \sum_{s \in S} y^+(s) - i \sum_{s \in S} y^-(s),$$

where

$$x^+(s) = \max(\mathrm{Re}[f(s)], 0), \quad x^-(s) = \max(-\mathrm{Re}[f(s)], 0),$$

$$y^+(s) = \max(\mathrm{Im}[f(s)], 0), \quad y^-(s) = \max(-\mathrm{Im}[f(s)], 0).$$

The following result follows immediately from this definition.

Lemma 6.6 *Let S be a countable set, and let $f\colon S \to \mathbb{C}$ be a summable function on S. Then*

$$\sum_{s \in S} f(s) = \sum_{s \in S} \mathrm{Re}[f(s)] + i \sum_{s \in S} \mathrm{Im}[f(s)].$$

Lemma 6.7 *Let S be a countable set, and let $f\colon S \to \mathbb{R}$ be a summable real-valued function on S. Suppose that $f(s) = p(s) - q(s)$ for all $s \in S$, where $p(s) \geq 0$ and $q(s) \geq 0$ for all $s \in S$. Suppose also that $\sum\limits_{s \in S} p(s) < +\infty$ and $\sum\limits_{s \in S} q(s) < +\infty$. Then*

$$\sum_{s \in S} f(s) = \sum_{s \in S} p(s) - \sum_{s \in S} q(s).$$

Proof It follows from the definition of $\sum_{s \in S} f(s)$ that

$$\sum_{s \in S} f(s) = \sum_{s \in S} x^+(s) - \sum_{s \in S} x^-(s),$$

where

$$x^+(s) = \max(\mathrm{Re}[f(s)], 0), \quad x^-(s) = \max(-\mathrm{Re}[f(s)], 0).$$

Then $f(s) = x^+(s) - x^-(s) = p(s) - q(s)$, and therefore $x^+(s) + q(s) = x^-(s) + p(s)$ for all $s \in S$. It follows from Lemma 6.5 that

$$\sum_{s \in S} x^+(s) + \sum_{s \in S} q(s) = \sum_{s \in S} x^-(s) + \sum_{s \in S} p(s).$$

Therefore

$$\sum_{s \in S} f(s) = \sum_{s \in S} x^+(s) - \sum_{s \in S} x^-(s) = \sum_{s \in S} p(s) - \sum_{s \in S} q(s),$$

as required. ∎

Proposition 6.8 *Let S be a countable set, and let $f\colon S \to \mathbb{C}$ and $g\colon S \to \mathbb{C}$ be summable functions on S. Then*

$$\sum_{s \in S} (f(s) + g(s)) = \sum_{s \in S} f(s) + \sum_{s \in S} g(s).$$

Proof Let

$$\begin{aligned} x^+(s) &= \max(\mathrm{Re}[f(s)], 0), \quad & x^-(s) &= \max(-\mathrm{Re}[f(s)], 0), \\ y^+(s) &= \max(\mathrm{Im}[f(s)], 0), \quad & y^-(s) &= \max(-\mathrm{Im}[f(s)], 0), \\ u^+(s) &= \max(\mathrm{Re}[g(s)], 0), \quad & u^-(s) &= \max(-\mathrm{Re}[g(s)], 0), \\ v^+(s) &= \max(\mathrm{Im}[g(s)], 0), \quad & v^-(s) &= \max(-\mathrm{Im}[g(s)], 0). \end{aligned}$$

Then

$$\sum_{s \in S} f(s) = \sum_{s \in S} x^+(s) - \sum_{s \in S} x^-(s) + i \sum_{s \in S} y^+(s) - i \sum_{s \in S} y^-(s)$$

and

$$\sum_{s \in S} g(s) = \sum_{s \in S} u^+(s) - \sum_{s \in S} u^-(s) + i \sum_{s \in S} v^+(s) - i \sum_{s \in S} y^-(s).$$

Now

$$\mathrm{Re}[f(s) + g(s)] = x^+(s) + u^+(s) - (x^-(s) + u^-(s))$$

for all $s \in S$. It follows from Lemma 6.5, Lemma 6.6 and Lemma 6.7 that

$$\begin{aligned} \mathrm{Re}\left[\sum_{s \in S} (f(s) + g(s))\right] &= \sum_{s \in S} \mathrm{Re}[f(s) + g(s)] \\ &= \sum_{s \in S} (x^+(s) + u^+(s)) - \sum_{s \in S} (x^-(s) + u^-(s)) \\ &= \sum_{s \in S} x^+(s) + \sum_{s \in S} u^+(s) - \sum_{s \in S} x^-(s) - \sum_{s \in S} u^-(s) \\ &= \sum_{s \in S} \mathrm{Re}[f(s)] + \sum_{s \in S} \mathrm{Re}[g(s)] \\ &= \mathrm{Re}\left[\sum_{s \in S} f(s) + \sum_{s \in S} g(s)\right]. \end{aligned}$$

Similarly

$$\operatorname{Im}\left[\sum_{s\in S}(f(s)+g(s))\right] = \operatorname{Im}\left[\sum_{s\in S} f(s) + \sum_{s\in S} g(s)\right].$$

Therefore

$$\sum_{s\in S}(f(s)+g(s)) = \sum_{s\in S} f(s) + \sum_{s\in S} g(s),$$

as required. ∎

Corollary 6.9 *Let S be a countable set, and let $f\colon S \to \mathbb{C}$ be a summable function on S. Then*

$$\sum_{s\in S} cf(s) = c\sum_{s\in S} f(s)$$

for all complex numbers c.

Proof Let $f(s) = x(s) + iy(s)$ for all $s \in S$, where $x(s) \in \mathbb{R}$ and $y(s) \in \mathbb{R}$, and let $c = a+ib$, where $a, b \in \mathbb{R}$. Then $cf(s) = ax(s)-by(s)+i(ay(s)+bx(s))$ for all $s \in S$. Now

$$\sum_{s\in S} x(s) = \sum_{s\in S} x^+(s) - \sum_{s\in S} x^-(s)$$

and

$$\sum_{s\in S} ax(s) = \sum_{s\in S} ax^+(s) - \sum_{s\in S} ax^-(s).$$

(This last identity follows directly from the definition of the sum of a real-valued function over a countable set, on considering separately the cases when $a \geq 0$ and when $a \leq 0$.) It follows that

$$\sum_{s\in S} ax(s) = a\sum_{s\in S} x(s).$$

Similarly

$$\sum_{s\in S} bx(s) = b\sum_{s\in S} x(s), \quad \sum_{s\in S} ay(s) = a\sum_{s\in S} y(s), \quad \sum_{s\in S} by(s) = b\sum_{s\in S} y(s).$$

It follows that

$$\begin{aligned}
\sum_{s\in S} cf(s) &= \sum_{s\in S} ax(s) - \sum_{s\in S} by(s) + i\left(\sum_{s\in S} ay(s) + \sum_{s\in S} bx(s)\right) \\
&= a\sum_{s\in S} x(s) - b\sum_{s\in S} y(s) + i\left(a\sum_{s\in S} y(s) + b\sum_{s\in S} x(s)\right) \\
&= (a+ib)\left(\sum_{s\in S} x(s) + i\sum_{s\in S} y(s)\right) = c\sum_{s\in S} f(s),
\end{aligned}$$

as required. $\blacksquare$

Corollary 6.10 *Let S be a countable set, and let $f\colon S \to \mathbb{C}$ be a summable function on S. Then*

$$\left|\sum_{s\in S} f(s)\right| \leq \sum_{s\in S} |f(s)|.$$

Proof There exists a complex number c satisfying $|c| = 1$ for which

$$\left|\sum_{s\in S} f(s)\right| = c\sum_{s\in S} f(s)$$

Then

$$\left|\sum_{s\in S} f(s)\right| = \sum_{s\in S} \mathrm{Re}[cf(s)] \leq \sum_{s\in S} |f(s)|,$$

as required. $\blacksquare$

Proposition 6.11 *Let $\mathcal{C}$ be a countable collection of countable sets, let V be the union $\bigcup_{S\in\mathcal{C}} S$ of the sets belonging to the collection $\mathcal{C}$, and let $f\colon V \to [0, +\infty]$ be a function on V taking values in the set $[0, +\infty]$ of non-negative extended real numbers. Suppose that the sets in the collection $\mathcal{C}$ are pairwise disjoint, and that either $\displaystyle\sum_{s\in V} |f(s)| < +\infty$ or $\displaystyle\sum_{S\in\mathcal{C}} \left(\sum_{s\in S} |f(s)|\right) < \infty$. Then*

$$\sum_{s\in V} f(s) = \sum_{S\in\mathcal{C}} \left(\sum_{s\in S} f(s)\right)$$

Proof We set

$$f(s) = x^+(s) - x^-(s) + iy^+(s) - iy^-(s),$$

for all $(s) \in S \times T$, where

$$x^+(s) = \max(\mathrm{Re}[f(s)], 0), \quad x^-(s) = \max(-\mathrm{Re}[f(s)], 0),$$

$$y^+(s) = \max(\mathrm{Im}[f(s)], 0), \quad y^-(s) = \max(-\mathrm{Im}[f(s)], 0).$$

It follows from Corollary 6.4 that

$$\sum_{s\in V} \lambda(s) = \sum_{S\in\mathcal{C}} \left(\sum_{s\in S} \lambda(s)\right)$$

in the cases when $\lambda(s) = |f(s)|$, $\lambda(s) = x^+(s)$, $\lambda(s) = x^-(s)$, $\lambda(s) = y^+(s)$ and $\lambda(s) = y^-(s)$. Thus if at least one of the quantities $\sum\limits_{s \in V} |f(s)|$ and $\sum\limits_{S \in C} \left(\sum_{s \in S} |f(s)|\right)$ is finite then

$$\sum_{s \in V} x^+(s) < +\infty, \quad \sum_{s \in V} x^-(s) < +\infty,$$

$$\sum_{s \in V} y^+(s) < +\infty \quad \text{and} \quad \sum_{s \in V} y^-(s) < +\infty.$$

But then

$$\sum_{v \in W} f(s) = \sum_{v \in W} x^+(s) - \sum_{v \in W} x^-(s) + i \sum_{v \in W} y^+(s) - i \sum_{v \in W} y^-(s)$$

for any subset W of V. The result therefore follows on applying Corollary 6.4 to the functions that send $s \in V$ to $x^+(s)$, $x^-(s)$, $y^+(s)$ and $y^-(s)$. ∎

The following result now follows directly from Proposition 6.11.

Corollary 6.12 *Let S and T be countable sets, and let $f: S \times T \to [0, +\infty]$ be a complex-valued function on $S \times T$. Suppose that at least one of the quantities*

$$\sum_{s \in S} \left(\sum_{t \in T} |f(s,t)| \right), \quad \sum_{(s,t) \in S \times T} |f(s,t)|, \quad \sum_{t \in T} \left(\sum_{s \in S} |f(s,t)| \right)$$

is finite. Then all these quantities are finite, and

$$\sum_{s \in S} \left(\sum_{t \in T} f(s,t) \right) = \sum_{(s,t) \in S \times T} f(s,t) = \sum_{t \in T} \left(\sum_{s \in S} f(s,t) \right).$$

Example The exponential function of complex analysis is defined such that $\exp z = \sum\limits_{n=0}^{+\infty} \dfrac{z^n}{n!}$ for all complex numbers z.

Let z and w be complex numbers, let P be the set of non-negative integers, and let $f: P \to P$ be the function defined such that

$$f(j,k) = \frac{z^j z^k}{j!k!}$$

for all non-negative integers j and k. Now

$$\begin{aligned}\sum_{j\in P}\left(\sum_{k\in P}|f(j,k)|\right) &= \sum_{j\in P}\left(\frac{|z|^j}{j!}\sum_{k\in P}\frac{|w|^k}{k!}\right) = \sum_{j\in P}\left(\frac{|z^j|}{j!}\exp|w|\right)\\ &= \exp|z|\exp|w|,\end{aligned}$$

Thus f is a summable function on $P \times P$. Moreover it follows from Corollary 6.12 that

$$\begin{aligned}\sum_{(j,k)\in P\times P} f(j,k) &= \sum_{j\in P}\left(\sum_{k\in P} f(j,k)\right) = \sum_{j\in P}\left(\frac{z^j}{j!}\sum_{k\in P}\frac{w^k}{k!}\right)\\ &= \sum_{j\in P}\left(\frac{z^j}{j!}\exp(w)\right) = \exp z\exp w.\end{aligned}$$

Now $P \times P$ is the disjoint union of the sets $D_0, D_1, D_2, D_3, \ldots$, where

$$D_n = \{(j,k) \in P \times P : j + k = n\}$$

for each non-negative integer n. It follows from Proposition 6.11 that

$$\sum_{(j,k)\in P\times P} f(j,k) = \sum_{n=0}^{+\infty}\left(\sum_{(j,k)\in D_n} f(j,k)\right).$$

But $k = n - j$ for all $(j,k) \in D_n$, and therefore

$$\begin{aligned}\sum_{(j,k)\in D_n} f(j,k) &= \sum_{j=0}^{n} f(j,n-j) = \sum_{j=0}^{n}\frac{z^j w^{n-j}}{j!k!} = \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j} z^j w^{n-j}\\ &= \frac{(z+w)^n}{n!}.\end{aligned}$$

Therefore

$$\begin{aligned}\exp z\exp w &= \sum_{(j,k)\in P\times P} f(j,k) = \sum_{n=0}^{+\infty}\left(\sum_{(j,k)\in D_n} f(j,k)\right) = \sum_{n=0}^{+\infty}\frac{(z+w)^n}{n!}\\ &= \exp(z+w).\end{aligned}$$

Thus this standard identity for the exponential function is a consequence of the basic theory of summable functions on countable sets developed above, as is the more general result involving Cauchy products of absolutely convergent infinite series.