Math 73/103 Assignment Three
Due TBA

CLARIFICATION: Let's review of notation and terminology. Lebesgue measure, $(\mathbf{R}, \mathfrak{M}, m)$, is the complete measure coming from the explicit outer measure m^* we defined in lecture. In particular, $\mathfrak{M}$ is the σ-algebra of all m^*-measurable sets. A Lebesgue measurable function $f : \mathbf{R} \to \mathbf{C}$ is just a function such that $f^{-1}(V) \in \mathfrak{M}$ for any open set $V \subset \mathbf{C}$. We say f is Borel if $f^{-1}(V)$ is a Borel set in $\mathbf{R}$ for every open set V. We say $f \in \mathcal{L}^1(\mathbf{R}, \mathfrak{M}, m)$, or that f is Lebesgue integrable, if f is measurable and $\int_{\mathbf{R}} |f| \, dm < \infty$. We have also used the notation $L^+(\mathbf{R}, \mathfrak{M}, m)$ for the collection Lebesgue measurable functions f such that $f \geq 0$ everywhere.

1. (We did this problem in lecture, so don't turn it in. I'm just including it for reference.) Recall that a sequence $\{f_n\}$ of measurable functions from $(X, \mathfrak{M})$ to $\mathbf{C}$ converges in measure to a measurable function $f : X \to \mathbf{C}$ if for all $\epsilon > 0$ we have $\lim_{n\to\infty} \mu\big(E_n(\epsilon)\big) = 0$ where

$$E_n(\epsilon) = \{\, x \in X : |f_n(x) - f(x)| \geq \epsilon \,\}.$$

Show that, as claimed in lecture, if $\{f_n\}$ converges to f in measure then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which converges to f almost everywhere.

Some suggestions:

(a) Let n_k be such that $n \geq n_k$ implies $\mu(E_n(2^{-k})) < 2^{-k}$.

(b) Let $E_k = E_{n_k}(2^{-k})$ and $G_k = \bigcup_{m \geq k} E_m$.

(c) Show that $f_{n_k}(x) \to f(x)$ if $x \notin A := \bigcap_{k=1}^{\infty} G_k$.

Another of Littlewood's Principles is that a pointwise convergent sequence of functions is nearly uniformly convergent. This is also known as "Egoroff's Theorem".

2. (We also proved this in lecture. So don't turn it in.) Prove Egoroff's Theorem: Suppose that $(X, \mathfrak{M}, \mu)$ is a finite measure space — that is, $\mu(X) < \infty$. Suppose that $\{f_n\}$ is a sequence of measurable functions converging almost everywhere to a measurable function $f : X \to \mathbf{C}$. Then for all $\epsilon > 0$ there is a set $E \in \mathfrak{M}$ such that $\mu(X \setminus E) < \epsilon$ and $f_n \to f$ uniformly on E.

Some suggestions:

(a) There is no harm in assuming that $f_n \to f$ everywhere.

(b) Let $E_n(k) = \bigcup_{m=n}^{\infty} \{\, x \in X : |f_m(x) - f(x)| \geq \frac{1}{k} \,\}$.

(c) Show that $\lim_{n\to\infty} \mu(E_n(k)) = 0$. (You need $\mu(X) < \infty$ here.)

(d) Fix $\epsilon > 0$ and k. Choose $n_k \geq n$ so that $\mu(E_{n_k}(k)) < \frac{\epsilon}{2^{-k}}$, and let

$$E = \bigcup_{k=1}^{\infty} E_{n_k}(k).$$

3. Suppose that $f \in \mathcal{L}^1(X, \mathfrak{M}, m)$ is a Lebesgue integrable function on the real line. Let $\epsilon > 0$. Show that there is a continuous function g that vanishes outside a bounded interval such that $\|f - g\|_1 < \epsilon$. (Hint: this is easy if f is the characteristic function of a bounded interval: draw a picture. Also invoke problem #8d from the second homework assignment.)

Fortunately, Littlewood only had three principles. The final one is that every measurable function is nearly continuous. This is known as "Lusin's Theorem".

4. Prove Lusin's Theorem: Suppose that f is a Lebesgue measurable function on $[a, b] \subset \mathbf{R}$. Given $\epsilon > 0$, show that there is a closed subset $K \subset [a, b]$ such that $m([a, b] \setminus K) < \epsilon$ and that $f|_K$ is continuous. (Suggestion use problem 3, Egoroff's Theorem and that fact that the uniform limit of continuous functions is continuous.)

5. Suppose that ρ is a premeasure on an algebra $\mathcal{A}$ of sets in X. Let ρ^* be the associated outer measure.

(a) Show that $\rho^*(E) = \rho(E)$ for all $E \in \mathcal{A}$.

(b) If $\mathfrak{M}^*$ is the σ-algebra of ρ^*-measurable sets, show that $\mathcal{A} \subset \mathfrak{M}^*$.

6. Let m be Lebesgue measure on $[0, 1]$ and let μ be counting measure. Clearly, $m \ll \mu$. Show that there is no function f satisfying the conclusion of the Radon-Nikodym Theorem. Why is this not a counter-example to the Radon-Nikodym Theorem.

7. Suppose that $f_n \to f$ in measure and that there is a $g \in \mathcal{L}^1(X, \mathfrak{M}, \mu)$ is such that $|f_n(x)| \leq g(x)$ for all $x \in X$. Show that $f_n \to f$ in $L^1(X, \mathfrak{M}, \mu)$.

8. [Optional: Do NOT turn in] Suppose that $f : [a, b] \to \mathbf{R}$ is a bounded function. We want to show that f is Riemann integrable if and only if

$$m\big(\{\, x \in [a, b] : f \text{ is not continuous at } x \,\}\big) = 0.$$

In [1, Theorem 2.28], Folland suggests the following strategy. Let

$$H(x) = \lim_{\delta \to 0}\big(\sup\{\, f(y) : |y - x| \le \delta \,\}\big) \quad \text{and} \quad h(x) = \lim_{\delta \to 0} \inf\{\, f(y) : |y - x| \le \delta \,\}.$$

(a) Show that f is continuous at x if and only if $H(x) = h(x)$.

(b) In the notation of our proof in lecture that Riemann integral functions are Lebesgue integrable, show that $h = \ell$ almost everywhere and $H = u$ almost everywhere.

(c) Conclude that $\int_a^b h\, dm = \mathcal{R}\underline{\int}_a^b f$ and $\int_a^b H\, dm = \mathcal{R}\overline{\int}_a^b f$.

References

[1] Gerald B. Folland, *Real analysis*, Second, John Wiley & Sons Inc., New York, 1999. Modern techniques and their applications, A Wiley-Interscience Publication. MR2000c:00001