Second Homework Assignment
Math 73/103
Due TBA.

1. Page 32 of the text, problem #6.[1] (Note that we have already shown that $\mathfrak{M}$ is a σ-algebra so there is no need to show it again. In the third edition, you are also asked to decribe the $\mathfrak{M}$-measurable functions and their integrals. The key is to notice that they must be constant μ-almost everywhere.)

2. Page 32 of the text, problem #7.

3. Page 32 of the text, problem #10.

4. Page 32 of the text, problem #12. (This is easy if f is bounded. If $E_n = \{\, x \in X : |f(x)| \le n \,\}$ and $f_n = \mathbb{I}_{E_n} \cdot f$, show that $\int_X |f - f_n| \, d\mu \to 0$.)

5. Suppose that Y is a topological space and that $\mathfrak{M}$ is a σ-algebra in Y containing all the Borel sets. Suppose in addition, μ is a measure on $(Y, \mathfrak{M})$ such that for all $E \in \mathfrak{M}$ we have

$$\mu(E) = \inf\{\, \mu(V) : V \text{ is open and } E \subset V \,\}. \tag{1}$$

Suppose also that

$$Y = \bigcup_{n=1}^{\infty} Y_n \quad \text{with } \mu(Y_n) < \infty \text{ for all } n \ge 1. \tag{2}$$

In \$25 words, μ is a *σ-finite outer regular measure* on $(Y, \mathfrak{M})$.

(a) Show that Lebesgue measure m is a σ-finite outer regular measure on $(\mathbf{R}, \mathfrak{M})$.

(b) Suppose E is a μ-measurable subset of Y.

(i) Given $\epsilon > 0$, show that there is an open set $V \subset Y$ and a closed set $F \subset Y$ such that $F \subset E \subset V$ and $\mu(V \setminus F) < \epsilon$.

(ii) Show that there is a G_δ-set $G \subset Y$ and a F_σ-set $A \subset Y$ such that $A \subset E \subset G$ and $\mu(G \setminus A) = 0$.

[1] "The text" means Rudin's *Real & Complex Analysis*, third edition.

(c) Argue that $(\mathbf{R}, \mathfrak{M}, m)$ is the completion of the restriction of Lebesgue measure to the Borel sets in $\mathbf{R}$.

6. Let m be Lebesgue measure on $\mathbf{R}$ and suppose that E is a set of finite measure. Given $\epsilon > 0$, show that there is a finite *disjoint* union F of open intervals such that $m(E \triangle F) < \epsilon$ where $E \triangle F := (E \setminus F) \cup (F \setminus E)$ is the symmetric difference. (This illustrates the first of Littlewood's three principles: "Every Lebesgue measurable set is nearly a disjoint union of open intervals".)

7. Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $(X, \mathfrak{M}_0, \mu_0)$ be its completion.

(a) If $f : X \to \mathbf{C}$ is μ_0-measurable, show that there is a μ-measurable function $g : X \to \mathbf{C}$ such that $f = g$ a.e. $[\mu_0]$.

(b) In part (a), is there necessarily a μ-null set N such that $f(x) = g(x)$ for all $x \notin N$?

(c) What does this result say about Lebesgue measurable functions and Borel functions on $\mathbf{R}$? (Compare with problem #14 on page 59 of the text.)

8. Suppose that $(X, \mathfrak{M}, \mu)$ is a measure space. Recall that $E \in \mathfrak{M}$ is called σ-finite if E is the countable union of sets of finite measure. Let $f \in \mathcal{L}^1(\mu)$.

(a) Show that $\{ x \in X : f(x) \neq 0 \}$ is σ-finite.

(b) Suppose that $f \geq 0$. Show that there are (measurable) simple functions φ_n such that $\varphi_n \nearrow f$ everywhere and there is a single σ-finite set outside of which the φ_n vanish.

(c) Given $\epsilon > 0$ show that there is simple function such that

$$\int_X |f - \varphi| \, d\mu < \epsilon.$$

(d) If $(X, \mathfrak{M}, \mu) = (\mathbf{R}, \mathfrak{M}, m)$ is Lebesgue measure, show that we can take the simple function φ in part (c) to be a step function — that is, a finite linear combination of characteristic functions of *intervals*.