Math 73/103 Final Exam

Instructions: You should return your exam to me in my office before noon on Tuesday, November 21, 2017.

(a) You must work alone. You may use the text, class notes and past homeworks, but no other sources allowed.

(b) Please turn in your solutions on one side only of $8\frac{1}{2}$" $\times$ 11" paper. Put name on the first page and staple in the upper left-hand corner.

(c) If you are not using LaTeX, please start each problem on a new page.

1. (25) Let u be a Harmonic function on a simply connected region Ω. Can u have any local maximums? What about local minimums? (Suggestion, if u were the real part of $f \in H(\Omega)$, then you could consider $g(z) = e^{f(z)}$.)

2. (25) Suppose that Ω is a region and that $a \in \Omega$.

(a) Suppose that $g \in H(\Omega \setminus \{a\})$ and that g has a pole of order m at a. Show that there is a $h \in H(\Omega)$ such that $h(a) \neq 0$ and

$$g(z) = \frac{h(z)}{(z-a)^m} \quad \text{for all } z \in \Omega \setminus \{a\}.$$

(b) Show that $f \in H(\Omega)$ has a zero of order m at a if and only if $1/f$ has a pole of order m at a.

3. (25) Suppose that Ω is a simply connected region and that $f \in H(\Omega)$ never vanishes. Show that there is a $h \in H(\Omega)$ such that $f = \exp(h)$. (This was part of our "End Game Theorem" which I now realize I never assigned.)

4. (25) Suppose that f has an isolated singularity at a and that the real part of f is bounded above near a; that is, there is a $r > 0$ such that

$$\operatorname{Re} f(z) \leq M < \infty \quad \text{for all } z \in D'_r(a). \tag{‡}$$

Show that a is a removable singularity for f. (This problem is quite a bit easier of we replace "$\operatorname{Re} f(z) \leq M$" with "$|\operatorname{Re} f(z)| \leq M$" in (‡). You can work that version for partial credit if you choose — just be clear which version you are solving. And just to be clear, no it is not "obvious" that if $g(z) = e^{f(z)}$ as a removable singularity at a, then so does f.)

5. (25) Let $(X, \mathfrak{M}, \mu)$ be a measure space with $\mu(X) = 1$. For each $n \in \mathbf{Z}_+$, let $A_n \in \mathfrak{M}$ be such that $\mu(A_n) = 1$. Show that if $A = \bigcap_n A_n$, then $\mu(A) = 1$.

6. (25) Let $(\mathbf{R}, \mathfrak{M}, m)$ be Lebesgue measure. Recall that $E \in \mathfrak{M}$ if and only if $E + y \in \mathfrak{M}$ for all $y \in \mathbf{R}$, and that $m(E) = m(E + y)$.

(a) Let $f \in \mathcal{L}^1(m)$ and $y \in \mathbf{R}$. Define $g(x) = f(x - y)$. Show that $g \in \mathcal{L}^1(m)$ and that

$$\int_{\mathbf{R}} f(x)\, dm(x) = \int_{\mathbf{R}} f(x - y)\, dm(x).$$

(b) If $f \in \mathcal{L}^1(m)$, let $\lambda_y(f) \in \mathcal{L}^1(m)$ be given by $\lambda_y(f)(x) = f(x - y)$. Show that $y \mapsto \lambda_y(f)$ is continuous from $\mathbf{R}$ to $L^1(m)$ in the sense that if $y_n \to y$ in $\mathbf{R}$, then $\|\lambda_{y_n}(f) - \lambda_y(f)\|_1 \to 0$.

(Hint: in part (a) start with characteristic functions. In part (b), you can reduce to the case where $y = 0$, and the conclusion is not so hard if f is continuous and vanishes off a bounded interval.)

7. (25) Recall that if X is a topological space, then $\mathfrak{B}(X)$ is the σ-algebra of Borel sets in X. Show that $\mathfrak{B}(\mathbf{R}^2) = \mathfrak{B}(\mathbf{R}) \otimes \mathfrak{B}(\mathbf{R})$.

8. (25) Suppose that $f : \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable.

(a) Show that $F : (\mathbf{R}^2, \mathfrak{M} \otimes \mathfrak{M}) \to (\mathbf{R}^2, \mathfrak{B}(\mathbf{R}^2))$ given by $F(x, y) = \big(f(x), y\big)$ is measurable. (This just means that $F^{-1}(V) \in \mathfrak{M} \otimes \mathfrak{M}$ when V is open in $\mathbf{R}^2$.)

(b) Show that

$$G(f) = \{\, \big(x, f(x)\big) \in \mathbf{R}^2 : x \in \mathbf{R} \,\}$$

is in $\mathfrak{M} \otimes \mathfrak{M}$.

(c) Show that for almost all y,

$$m\big(\{\, x \in \mathbf{R} : f(x) = y \,\}\big) = 0.$$

(Hint: all these parts are connected. Any if you were to use something like Tonelli or Fubini's Theorem, you should carefully explain how.)