NAME: GRADE:

Math 73/103

Midterm Examination: Measure Theory

1. Let X be a measure space and $f : X \longrightarrow \mathbb{R}$ a measurable function. If $A > 0$ is a real number, define f_A by

$$f_A(x) = \begin{cases} -A & \text{if } f(x) < -A \\ f(x) & \text{if } |f(x)| \leq A \\ A & \text{if } f(x) > A \end{cases}$$

a. Draw a picture.

b. Is f_A measurable?

2. Let $(X, \mathfrak{M}, \mu)$ be a measured space, $\mathfrak{N}$ a σ-algebra contained in $\mathfrak{M}$ and ν the restriction of μ to $\mathfrak{N}$. Assume that f is a non-negative ν-measurable function.

a. Prove that f is μ-measurable.

b. Compare $\int_X f\,d\nu$ and $\int_X f\,d\mu$. Can these integrals differ?

3. Let $\mathfrak{M}$ be the σ-algebra on $\mathbb{R}$ generated by the singletons.

a. Describe $\mathfrak{M}$.

b. Prove that the function $x \mapsto (x, |x|)$ is measurable from $(\mathbb{R}, \mathfrak{M})$ to $(\mathbb{R}^2, \mathfrak{M} \otimes \mathfrak{M})$.

c. Is the diagonal $\Delta = \{(x, x) \ : \ x \in \mathbb{R}\}$ an element of $\mathfrak{M} \otimes \mathfrak{M}$?

4. Let $(X, \mathfrak{M}, \mu)$ be a measured space such that $\mu(X) < \infty$. Let $g : \mathbb{C} \to \mathbb{C}$ be a continuous function and assume the existence of constants $A, B > 0$ such that

$$|g(z)| \leq A|z| + B$$

for all $z \in \mathbb{C}$.

a. Let $v \in \mathscr{L}^1(X, \mathfrak{M}, \mu)$. Prove that $g \circ v \in \mathscr{L}^1(X, \mathfrak{M}, \mu)$.

For $u \in L^1(X, \mathfrak{M}, \mu)$, denote by $G(u)$ the class in $L^1(X, \mathfrak{M}, \mu)$ of $g \circ v$, where $v \in u$.

b. Verify that $G(u)$ is well-defined, that is, it does not depend on the choice of v.

c. Let $\{u_n\}_{n\geq 1}$ be a sequence in $L^1(X, \mathfrak{M}, \mu)$. Suppose that $u_n \xrightarrow{\text{a.e.}} u$ and all the sets

$$E_n = \left\{x \in X \ : \ |u_n(x)| > \sqrt{2}\right\}$$

have measure 0. Prove that $G(u_n) \xrightarrow{L^1} G(u)$.

d. (Optional) Prove that $G : L^1(X, \mathfrak{M}, \mu) \longrightarrow L^1(X, \mathfrak{M}, \mu)$ is continuous.

5. Let f be a Lebesgue integrable function on $\mathbb{R}$. For $a > 0$, let $f_\alpha : x \mapsto f(ax)$.

Express $\displaystyle\int_{\mathbb{R}} f_a \, d\lambda$ in terms of $\displaystyle\int_{\mathbb{R}} f \, d\lambda$.

6. Let ρ be the premeasure defined on the algebra $\mathcal{A} = \{\emptyset\,,\ [1,2]\,,\ [1,2]^c\,,\ \mathbb{R}\}$ by

$$\rho([1,2]) = 1 \qquad \text{and} \qquad \rho([1,2]^c) = +\infty.$$

a. Determine the outer measure ρ^* induced by ρ.

b. Describe the ρ^*-measurable subsets of $\mathbb{R}$.

[EXTRA SPACE - p.1]

[EXTRA SPACE - p.2]