# Math 73/103 - Fall 2016
# Problem set 4

1. Recall that a sequence $\{f_n\}$ of measurable functions from $(X, \mathfrak{M})$ to $\mathbf{C}$ converges in measure to a measurable function $f : X \to \mathbf{C}$ if for all $\varepsilon > 0$ we have $\lim_{n\to\infty} \mu\big(E_n(\varepsilon)\big) = 0$ where

$$E_n(\varepsilon) = \{\, x \in X : |f_n(x) - f(x)| \geq \varepsilon \,\}.$$

Show that, as claimed in lecture, if $\{f_n\}$ converges to $f$ in measure then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which converges to $f$ almost everywhere.

Some suggestions:

(a) Let $n_k$ be such that $n \geq n_k$ implies $\mu(E_n(2^{-k})) < 2^{-k}$.

(b) Let $E_k = E_{n_k}(2^{-k})$ and $G_k = \bigcup_{m\geq k} E_m$.

(c) Show that $f_{n_k}(x) \to f(x)$ if $x \notin A := \bigcap_{k=1}^{\infty} G_k$.

The last of Littlewood's three principles is that every measurable function is nearly continuous. This is known as "Lusin's Theorem".

2. Prove Lusin's Theorem: Suppose that $f$ is a Lebesgue measurable function on $[a, b] \subset \mathbf{R}$. Given $\varepsilon > 0$, show that there is a closed subset $K \subset [a, b]$ such that $\lambda([a, b] \setminus K) < \varepsilon$ and that $f|_K$ is continuous. (Suggestion: use Problem 1, Egoroff's Theorem and the fact that uniform limits of continuous functions are continuous.)

3. Let $\lambda$ be Lebesgue's measure on $[0, 1]$ and let $\mu$ be counting measure. Clearly, $\lambda \ll \mu$. Show that there is no function $f$ satisfying the conclusion of the Radon-Nikodym Theorem. Why is this not a counter-example to the Radon-Nikodym Theorem?

4. Suppose that $f_n \to f$ in measure and that there is a $g \in \mathcal{L}^1(X, \mathfrak{M}, \mu)$ is such that $|f_n(x)| \leq g(x)$ for all $x \in X$. Show that $f_n \to f$ in $L^1(X, \mathfrak{M}, \mu)$.

5. Let $\nu$ be a complex measure on $(X, \mathfrak{M})$.

(a) Show that there is a measure $\mu$ and a measurable function $\varphi : X \to \mathbf{C}$ so that $|\varphi| = 1$, and such that for all $E \in \mathfrak{M}$,

$$\nu(E) = \int_E \varphi \, d\mu. \tag{\dagger}$$

(Hint: write $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ for measures $\nu_i$. Put $\mu_0 = \nu_1 + \nu_2 + \nu_3 + \nu_4$. Then $\mu_0$ will satisfy (†) provided we don't require $|\varphi| = 1$. You can then use without proof the fact that any complex-valued measurable function $h$ can be written as $h = \varphi \cdot |h|$ with $\varphi$ unimodular and measurable.)

(b) Show that the measure $\mu$ above is unique, and that $\varphi$ is determined almost everywhere $[\mu]$. (Hint: if $\mu'$ and $\varphi'$ also satisfy (†), then show that $\mu' \ll \mu$, and that $\frac{d\mu'}{d\mu} = 1$ a.e. Also note that if $\varphi'$ is unimodular and $E \in \mathfrak{M}$, then $E = \bigcup_{i=1}^4 E_i$ where $E_1 = \{ x \in E : \operatorname{Re} \varphi' > 0 \}$, $E_2 = \{ x \in E : \operatorname{Re} \varphi' < 0 \}$, $E_3 = \{ x \in E : \operatorname{Im} \varphi' > 0 \}$, and $E_4 = \{ x \in E : \operatorname{Im} \varphi' < 0 \}$.)

Comment: the measure $\mu$ in question 5 is called the *total variation* of $\nu$, and the usual notation is $|\nu|$. It is defined by different methods in Rudin's text: see Chapter 6. One can prove facts like $|\nu|(E) \geq |\nu(E)|$ (although one doesn't always have $|\nu|(E) = |\nu(E)|$).

6. Suppose that $f : [a, b] \to \mathbf{R}$ is a bounded function. We want to show that $f$ is Riemann integrable if and only if

$$\lambda\bigl(\{ x \in [a, b] : f \text{ is not continuous at } x \}\bigr) = 0.$$

Folland suggests the following strategy. Let

$$H(x) = \lim_{\delta \to 0} \bigl(\sup\{ f(y) : |y - x| \leq \delta \}\bigr) \quad \text{and} \quad h(x) = \lim_{\delta \to 0} \inf\{ f(y) : |y - x| \leq \delta \}.$$

(a) Show that $f$ is continuous at $x$ if and only if $H(x) = h(x)$.

(b) In the notation of our proof that Riemann integral functions are Lebesgue integrable, show that $h = \ell$ almost everywhere and $H = u$ almost everywhere.

(c) Conclude that $\int_a^b h \, d\lambda = \mathcal{R}\underline{\int}_a^b f$ and $\int_a^b H \, d\lambda = \mathcal{R}\overline{\int}_a^b f$.