Math 73/103 Assignment Five
Due Monday, November 17, 2014

1. Note that every nonzero complex number z can be written in *polar form*: $z = re^{i\theta}$ where $r = |z|$ and θ is called an argument of z. Of course "the" *argument of* z is only defined up to a multiple of 2π.

(a) Show that $e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$.

(b) Conclude that if $z, w \in \mathbf{C}$, then $|zw| = |z||w|$, and that

(c) $e^z e^w = e^{z+w}$.

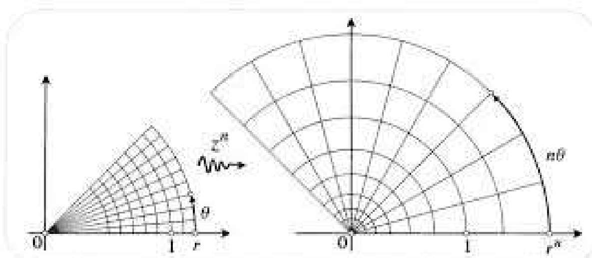

Figure 1: You can use problem 1 to visualize complex multiplication

2. Let Ω be a domain. Show that $f_n \to f$ uniformly on compact subsets of Ω if and only if $f_n \to f$ uniformly on every closed disk contained in Ω.

3. Prove the "Walking the Dog Lemma": Let γ_0 and γ_1 be closed paths. Let $a \in \mathbf{C}$ and suppose that

$$|\gamma_1(t) - \gamma_0(t)| < |a - \gamma_0(t)| \quad \text{for } t \in [0, 1].$$

Conclude that $\text{Ind}_{\gamma_0}(a) = \text{Ind}_{\gamma_1}(a)$. In other words, γ_0 and γ_1 wrap around a exactly the same number of times. (So if I walk Willy in a park with a lamp post in the center and wisely never come nearer the lamp post than the length of the leash, we both circle the lamp post the same number of times.)
(I suggest the following approach. Note that $a \notin \gamma_k$ for $k = 0, 1$. Let $\gamma(t) = \frac{\gamma_1(t)-a}{\gamma_0(t)-a}$. Observe that $\gamma^* \subset D = B_1(1)$, and conclude that $\text{Ind}_\gamma(0) = 0$.)

4. Prove Roche's Theorem: Suppose that f and g are analytic on and inside a simple closed contour Γ, and that for $z \in \Gamma$, $|f(z) - g(z)| < |f(z)|$. (Notice that this implies neither f nor g has zeros on Γ.) Show that $N_f = N_g$, where N_f is the number of zeros of f inside Γ counted up to multiplicity. (Use the Walking the Dog Lemma and the observation $N_f = \mathrm{Ind}_{f(\Gamma)}(0)$.)

Also, work problems 2, 3, 4, 5, 13 and 20 on pages 227–230 of the text.

- For problem 2: The Baire Category Theorem implies that if $\mathbf{C} = \bigcup F_n$ when each F_n closed, then some F_n has interior.
- For problem 4: Estimate $|f^{k+1}(z)|$.
- For problem 5: The hypotheses of the problem don't allow us to conclude even that the limit function $f = \lim_n f_n$ is continuous. Instead, you'll have to prove that $\{ f_n \}$ is uniformly Cauchy. You may want to use (and prove) that if $g_n(z) \leq M$ for all $z \in \gamma^*$ and g_n converges pointwise to a 0, then

$$\int_\gamma g(z)\, dz \to 0.$$

- For problem 20: Notice that $f_n' \to f'$ uniformly on compact subsets of Ω. Show this implies $f_n'/f_n \to f'/f$ uniformly on any γ^* provided $f \neq 0$ on γ^*.