Math 73/103 Assignment Three
Due Date TBA

CLARIFICATION: Let's review of notation and terminology. Lebesgue measure, $(\mathbf{R}, \mathfrak{M}, m)$, is the complete measure coming from the explicit outer measure m^* we defined in lecture. In particular, $\mathfrak{M}$ is the σ-algebra of all m^*-measurable sets. A Lebesgue measurable function $f : \mathbf{R} \to \mathbf{C}$ is just a function such that $f^{-1}(V) \in \mathfrak{M}$ for any open set $V \subset \mathbf{C}$. We say f is Borel if $f^{-1}(V)$ is a Borel set in $\mathbf{R}$ for every open set V. We say $f \in \mathcal{L}^1(\mathbf{R}, \mathfrak{M}, m)$, or that f is Lebesgue integrable, if f is measurable and $\int_{\mathbf{R}} |f| \, dm < \infty$. We have also used the notation $L^+(\mathbf{R}, \mathfrak{M}, m)$ for the collection Lebesgue measurable functions f such that $f \geq 0$ everywhere.

1. Suppose that $f \in \mathcal{L}^1(X, \mathfrak{M}, m)$ is a Lebesgue integrable function on the real line. Let $\epsilon > 0$. Show that there is a continuous function g that vanishes outside a bounded interval such that $\|f - g\|_1 < \epsilon$. (Hint: this is easy if f is the characteristic function of a bounded interval: draw a picture.)

Another of Littlewood's Principles is that a pointwise convergent sequence of functions is nearly uniformly convergent. This is also known as "Egoroff's Theorem".

2. Prove Egoroff's Theorem: Suppose that $(X, \mathfrak{M}, \mu)$ is a finite measure space — that is, $\mu(X) < \infty$. Suppose that $\{f_n\}$ is a sequence of measurable functions converging pointwise to a measurable function $f : X \to \mathbf{C}$. Then for all $\epsilon > 0$ there is a set $E \in \mathfrak{M}$ such that $\mu(X \setminus E) < \epsilon$ and $f_n \to f$ uniformly on E.

Some suggestions:

(a) There is no harm in assuming that $f_n \to f$ everywhere.

(b) Let $E_n(k) = \bigcup_{m=n}^{\infty} \{ x \in X : |f_m(x) - f(x)| \geq \frac{1}{k} \}$.

(c) Show that $\lim_{n\to\infty} \mu(E_n(k)) = 0$. (You need $\mu(X) < \infty$ here.)

(d) Fix $\epsilon > 0$ and k. Choose $n_k \geq n$ so that $\mu(E_{n_k}(k)) < \frac{\epsilon}{2^{-k}}$, and let

$$E = \bigcup_{k=1}^{\infty} E_{n_k}(k).$$

Fortunately, Littlewood only had three principles. The final one is that every measurable function is nearly continuous. This is known as "Lusin's Theorem".

3. Prove Lusin's Theorem: Suppose that f is a Lebesgue measurable function on $[a,b] \subset \mathbf{R}$. Given $\epsilon > 0$, show that there is a closed subset $K \subset [a,b]$ such that $m([a,b] \setminus K) < \epsilon$ and that $f|_K$ is continuous. (Suggestion use problem 1, Egoroff's Theorem and that fact that the uniform limit of continuous functions is continuous.)

4. Recall that a sequence $\{f_n\}$ of measurable functions from $(X, \mathfrak{M})$ to $\mathbf{C}$ converges in measure to a measurable function $f : X \to \mathbf{C}$ if for all $\epsilon > 0$ we have $\lim_{n\to\infty} \mu\big(E_n(\epsilon)\big) = 0$ where

$$E_n(\epsilon) = \{\, x \in X : |f_n(x) - f(x)| \geq \epsilon \,\}.$$

Show that, as claimed in lecture, if $\{f_n\}$ converges to f in measure then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which converges to f almost everywhere.

Some suggestions:

(a) Let n_k be such that $n \geq n_k$ implies $\mu(E_n(2^{-k})) < 2^{-k}$.

(b) Let $E_k = E_{n_k}(2^{-k})$ and $G_k = \bigcup_{m \geq k} E_m$.

(c) Show that $f_{n_k}(x) \to f(x)$ if $x \notin A := \bigcap_{k=1}^{\infty} G_k$.

5. Suppose that ρ is a premeasure on an algebra $\mathcal{A}$ of sets in X. Let ρ^* be the associated outer measure.

(a) Show that $\rho^*(E) = \rho(E)$ for all $E \in \mathcal{A}$.

(b) If $\mathfrak{M}^*$ is the σ-algebra of ρ^*-measurable sets, show that $\mathcal{A} \subset \mathfrak{M}^*$.

6. Let m be Lebesgue measure on $[0,1]$ and let μ be counting measure. Clearly, $m \ll \mu$. Show that there is no function f satisfying the conclusion of the Radon-Nikodym Theorem. Why is this not a counter-example to the Radon-Nikodym Theorem.

7. Suppose that $f_n \to f$ in measure and that there is a $g \in \mathcal{L}^1(X, \mathfrak{M}, \mu)$ is such that $|f_n(x)| \leq g(x)$ for all $x \in X$. Show that $f_n \to f$ in $L^1(X, \mathfrak{M}, \mu)$.

8. [Optional: Do NOT turn in] Suppose that $f : [a, b] \to \mathbf{R}$ is a bounded function. We want to show that f is Riemann integrable if and only if

$$m\big(\{\, x \in [a, b] : f \text{ is not continuous at } x \,\}\big) = 0.$$

In [1, Theorem 2.28], Folland suggests the following strategy. Let

$$H(x) = \lim_{\delta \to 0} \big(\sup\{\, f(y) : |y - x| \le \delta \,\}\big) \quad \text{and} \quad h(x) = \lim_{\delta \to 0} \inf\{\, f(y) : |y - x| \le \delta \,\}.$$

(a) Show that f is continuous at x if and only if $H(x) = h(x)$.

(b) In the notation of our proof in lecture that Riemann integral functions are Lebesgue integrable, show that $h = \ell$ almost everywhere and $H = u$ almost everywhere.

(c) Conclude that $\int_a^b h \, dm = \mathcal{R}\underline{\int}_a^b f$ and $\int_a^b H \, dm = \mathcal{R}\overline{\int}_a^b f$.

References

[1] Gerald B. Folland, *Real analysis*, Second, John Wiley & Sons Inc., New York, 1999. Modern techniques and their applications, A Wiley-Interscience Publication. MR2000c:00001