

**Homework for Math 103**  
**Assignment One – Due September 24**

1. Show that the *countable* union of sets of measure zero in  $\mathbf{R}$  has measure zero.
2. Suppose  $f : [a, b] \rightarrow \mathbf{R}$  is bounded, and let  $\mathcal{P}$  and  $\mathcal{Q}$  be partitions of  $[a, b]$ . Prove that  $L(f, \mathcal{P}) \leq U(f, \mathcal{Q})$ , where  $L(f, \mathcal{P})$  and  $U(f, \mathcal{Q})$  are the lower and upper Riemann sums, respectively, for  $f$  on  $[a, b]$ . (Hint: the result is trivial if  $\mathcal{P} = \mathcal{Q}$ ; now let  $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$ .)
3. Prove that a bounded function  $f : [a, b] \rightarrow \mathbf{R}$  is Riemann integrable on  $[a, b]$  if and only if for all  $\epsilon > 0$  there is a partition  $\mathcal{P}$  of  $[a, b]$  such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

4. (*Rudin*: page 31 #1) Suppose that  $(X, \mathcal{M})$  is a measurable space. Show that if  $\mathcal{M}$  is countable, then  $\mathcal{M}$  is finite. (Hint: since  $\mathcal{M}$  is countable, you can show that  $\omega_x = \bigcap \{ E : E \in \mathcal{M} \text{ and } x \in E \}$  belongs to  $\mathcal{M}$ . The sets  $\{\omega_x\}_{x \in X}$  partition  $X$ .)
5. Let  $X$  be an uncountable set and let  $\mathcal{M}$  be the collection of subsets  $E$  of  $X$  such that either  $E$  or  $E^c$  is countable. Prove that  $\mathcal{M}$  is a  $\sigma$ -algebra.
6. Recall from calculus that if  $\{a_n\}$  is a sequence of nonnegative real numbers, then  $\sum_{n=1}^{\infty} a_n = \sup_n s_n$ , where  $s_n = a_1 + \cdots + a_n$ . (Note the value  $\infty$  is allowed.)
  - (a) Show that  $\sum_{n=1}^{\infty} a_n = \sup\{\sum_{k \in F} a_k : F \text{ is a finite subset of } \mathbf{Z}^+ = 1, 2, 3, \dots\}$ . (The point of this problem is that if  $I$  is a (not necessarily countable) set, and if  $a_i \geq 0$  for all  $i \in I$ , then we can define  $\sum_{i \in I} a_i = \sup\{\sum_{k \in F} a_k : F \text{ is a finite subset of } I\}$ , and our new definition coincides with the usual one when both make sense.)
  - (b) Now let  $X$  be a set and  $f : X \rightarrow [0, \infty)$  a function. For each  $E \subset X$ , define

$$\nu(E) := \sum_{x \in E} f(x).$$

Show that  $\nu$  is a measure on  $(X, \mathcal{P}(X))$ . (In lecture, we considered the special cases of *counting measure*, where  $f(x) = 1$  for all  $x \in X$ , and the *delta measure at  $x_0$* , where  $f(x_0) = 1$  for some  $x_0 \in X$  and  $f(x) = 0$  otherwise. Another important example is the case where  $\sum_{x \in X} f(x) = 1$ . Then  $f$  is a (discrete) probability distribution on  $X$  and  $\nu(E)$  is the probability of the event  $E$  for this distribution.)

- (c) Let  $X$ ,  $f$ , and  $\nu$  be as in part (b). Show that if  $\nu(E) < \infty$ , then  $\{x \in E : f(x) > 0\}$  is countable. Hint: if  $\{x \in E : f(x) > 0\}$  is uncountable, then for some  $m \in \mathbf{Z}^+$ , the set  $\{x \in E : f(x) > \frac{1}{m}\}$  is infinite. (Note that this last result says that discrete probability distributions “live on” countable sample spaces.)

7. (*Rudin*: page 31 #3) Prove that if  $f$  is a real-valued function on a measurable space  $(X, \mathcal{M})$  such that  $\{x : f(x) \geq r\}$  is measurable for all rational  $r$ , then  $f$  is measurable.

8. (*Rudin*: page 31 #5) Suppose that  $f, g : (X, \mathcal{M}) \rightarrow [-\infty, \infty]$  are measurable functions. Prove that the sets

$$\{x : f(x) < g(x)\} \quad \text{and} \quad \{x : f(x) = g(x)\}$$

are measurable. (Remark: if  $h = f - g$  were defined, then this problem would be much easier (why?). The problem is that  $\infty - \infty$  and  $-\infty + \infty$  make no sense, so  $h$  may not be everywhere defined.)