How Many Borel Sets are There?

Object. *This series of exercises is designed to lead to the conclusion that if* $\mathcal{B}_{\mathbf{R}}$ *is the* σ*-algebra of Borel sets in* $\mathbf{R}$*, then*

$$\operatorname{Card}(\mathcal{B}_{\mathbf{R}}) = \mathfrak{c} := \operatorname{Card}(\mathbf{R}).$$

This is the conclusion of problem 4. As a bonus, we also get some insight into the "structure" of $\mathcal{B}_{\mathbf{R}}$ *via problem 2. This just scratches the surface. If you still have an itch after all this, you want to talk to a set theorist. This treatment is based on the discussion surrounding* [1, Proposition 1.23] *and* [2, Chap. V §10 #31].

For these problems, you will need to know a bit about well-ordered sets and transfinite induction. I suggest [1, §0.4] where transfinite induction is [1, Proposition 0.15]. Note that by [1, Proposition 0.18], there is an uncountable *well ordered* set Ω such that for all $x \in \Omega$, $I_x := \{\, y \in \Omega : y < x \,\}$ is countable. The elements of Ω are called the *countable ordinals*. We let $1 := \inf \Omega$. If $x \in \Omega$, then $x + 1 := \inf\{\, y \in \Omega : y > x \,\}$ is called the *immediate successor* of x. If there is a $z \in \Omega$ such that $z + 1 = x$, then z is called the *immediate predecessor* of x. If x has no immediate predecessor, then x is called a *limit ordinal*.[1]

1. Show that $\operatorname{Card}(\Omega) \le \mathfrak{c}$. (This follows from [1, Propositions 0.17 and 0.18]. Alternatively, you can use transfinite induction to construct an injective function $f : \Omega \to \mathbf{R}$.)[2]

2. If X is a set, let $\mathscr{P}(X)$ be the set of subsets of X — i.e., $\mathscr{P}(X)$ is the *power set* of X. Let $\mathscr{E} \subset \mathscr{P}(X)$. The object of this problem is to give a "concrete" description of the σ-algebra $\mathscr{M}(\mathscr{E})$ generated by $\mathscr{E}$. (Of course, we are aiming at describing the Borel sets in $\mathbf{R}$ which are generated by the collection $\mathscr{E}$ of open intervals.) For convenience, we assume that $\emptyset \in \mathscr{E}$.

Let

$$\mathscr{E}^c := \{\, E^c : E \in \mathscr{E} \,\} \quad \text{and} \quad \mathscr{E}_\sigma = \{\, \bigcup_{i=1}^{\infty} E_i : E_i \in \mathscr{E} \,\}.$$

(Note, I just mean that $\mathscr{E}_\sigma$ is the set of sets formed from countable unions of elements of $\mathscr{E}$. Since $\emptyset \in \mathscr{E}$, $\mathscr{E} \subset \mathscr{E}_\sigma$.)

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[1] The set of countable ordinals has a rich structure. We let $2 := 1 + 1$, and so on. The set $\{\, n \in \mathbf{N} \,\} \subset \Omega$ is countable, and so has a supremum ω (see [1, Proposition 0.19]). Then there are ordinals $\omega + 1$, $\omega + 2$, $\dots$, 2ω, $2\omega + 1, \dots$, ω^2, $\omega^2 + 1$, $\dots$, ω^ω, and so on.

[2] The issue of whether or not $\operatorname{Card}(\Omega) = \mathfrak{c}$ is the *continuium hypothesis*, and so is independent of the usual (ZFC) axioms of set theory.

We let $\mathscr{F}_1 := \mathscr{E} \cup \mathscr{E}^c$. If $x \in \Omega$, and if x has an immediate predecessor y, then we set

$$\mathscr{F}_x := (\mathscr{F}_y)_\sigma \cup \big((\mathscr{F}_y)_\sigma\big)^c.$$

If x is a limit ordinal, then we set

$$\mathscr{F}_x := \bigcup_{y<x} \mathscr{F}_y.$$

We aim to show that

$$\mathscr{M}(\mathscr{E}) = \bigcup_{x\in\Omega} \mathscr{F}_x \tag{$\dagger$}$$

(a) Observe that $\mathscr{F}_1 \subset \mathscr{M}(\mathscr{E})$.

(b) Show that if $F_y \subset \mathscr{M}(\mathscr{E})$ for all $y < x$, then $F_x \subset \mathscr{M}(\mathscr{E})$. Then use transfinite induction to conclude that $\mathscr{F}_x \subset \mathscr{M}(\mathscr{E})$ for all $x \in \Omega$.

(c) Show that the right-hand side of ($\dagger$) is closed under countable unions.

(d) Conclude that $\bigcup_{x\in\Omega} \mathscr{F}_x$ is a σ-algebra, and that ($\dagger$) holds.

3. Recall that if A and B are sets, then $\prod_{a\in A} B$ is simply the set of functions from A to B. For reasons that are unclear to me, this set is usually written B^A. Notice that $\prod_{i=1}^\infty B = \prod_{i\in\mathbf{N}} B$ is just the collection of sequences in B. Notice also that $\mathrm{Card}(B^A)$ depends only on $\mathrm{Card}(A)$ and $\mathrm{Card}(B)$.

(a) Check that

$$\prod_{i=1}^\infty \Big(\prod_{j=1}^\infty B\Big) = \prod_{(i,j)\in\mathbf{N}\times\mathbf{N}} B. \tag{$*$}$$

Thus the cardinality of either side of ($*$) is the same as $\prod_{i=1}^\infty B$.

(b) Use these observations together with the fact that $\mathrm{Card}\big(\prod_{i=1}^\infty\{\,0,1\,\}\big) = \mathfrak{c} := \mathrm{Card}(\mathbf{R})$ (which follows from [1, Proposition 0.12]) to show that

$$\mathrm{Card}\Big(\prod_{i=1}^\infty \mathbf{R}\Big) = \mathfrak{c}.$$

(c) Show that if $\mathrm{Card}(\mathscr{E}) = \mathfrak{c}$, then $\mathrm{Card}(\mathscr{E}_\sigma) = \mathfrak{c}$.

4. Let $\mathcal{B}_{\mathbf{R}}$ be the σ-algebra of Borel sets in $\mathbf{R}$. In [1, Proposition 0.14(b)], it is shown that if $\operatorname{Card}(A) \leq \mathfrak{c}$ and if $\operatorname{Card}(Y_x) \leq \mathfrak{c}$ for all $x \in A$, then $\bigcup_{x \in A} Y_x$ has cardinality bounded by $\mathfrak{c}$. By following the given steps, use this observation, as well as problems 2 and 3, to show that

$$\operatorname{Card}(\mathcal{B}_{\mathbf{R}}) = \mathfrak{c}. \tag{$\ddagger$}$$

(a) Let $\mathscr{E}$ be the collection of open intervals (including the empty set) in $\mathbf{R}$. Then $\operatorname{Card}(\mathscr{E}) = \mathfrak{c}$.

(b) $\mathcal{B}_{\mathbf{R}} = \mathscr{M}(\mathscr{E})$.

(c) Define $\mathscr{F}_x$ as in problem 2. Use transfinite induction and problem 3 to prove that $\operatorname{Card}(F_x) = \mathfrak{c}$ for all $x \in \Omega$.

(d) Use problem 2 to conclude that $\mathscr{M}(\mathscr{E}) = \mathcal{B}_{\mathbf{R}}$ has the cardinality claimed in ($\ddagger$).[4]

References

[1] Gerald B. Folland, *Real analysis*, Second, John Wiley & Sons Inc., New York, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[2] Anthony W. Knapp, *Basic real analysis*, Cornerstones, Birkhäuser Boston Inc., Boston, MA, 2005. Along with a companion volume *Advanced real analysis*.

[4] It is my ***understanding*** that the classes $\mathscr{F}_x$ are all distinct; that is, $\mathscr{F}_x \subsetneq \mathscr{F}_y$ if $x < y$ in Ω. But I don't have a reference or a proof at hand.