Final, Math 103, Fall 06

Marius Ionescu

Due: **Sunday,** December 3, Noon

The first five problems are from our textbook. There is also a bounus problem, which can replace any of the other. Good luck!

As we have discussed, the graduate students have to do all the problems (with the exception of the bonus problem). The undergraduate students are required to solve only the problems 1,2,3,6,7,8. Everybody is welcome to try all the problems.

1. Problem 27 on page 60.
2. Problem 50 on page 69.
3. Problem 3 on page 88.
4. Problem 12 on page 92.
5. Problem 13 on page 92.
6. Prove that $\int_0^1 \int_1^\infty (e^{-xy} - 2e^{-2xy}) \mathrm{d}x \mathrm{d}y \neq \int_1^\infty \int_0^1 (e^{-xy} - 2e^{-2xy}) \mathrm{d}y \mathrm{d}x$. Why are they unequal?
7. Let $(X, \mathcal{M}, \mu)$ be a finite, complete measure space. Let m be the Lebesque measure on $I = [0,1]$, and $f : X \to I$, a measurable function. Show that $B = \{(x, f(x)) \ : \ x \in X\}$ is measurable in the product σ-algebra $\mathcal{M} \otimes \mathcal{B}(I)$. Prove that $\mu \otimes m(B) = 0$.
8. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $\{f_n\} \subset L^1(\mu)$ and $f \in L^1(\mu)$ such that $|f_n(x)| \leq |f(x)|$ for all $x \in X$. Prove that
$$\int \liminf f_n \leq \liminf \int f_n \leq \limsup \int f_n \leq \int \limsup f_n.$$
9. Let $([0,1], \mathcal{M}, m)$ be the Lebesque measure, and f_n, f be Lebesque integrable functions on $[0.1]$ such that $f_n \to f$ a.e. on $[0,1]$ and $\int_0^1 |f_n| \to \int_0^1 |f|$.
   (a) Prove that $f_n \to f$ in $L^1([0,1])$.
   (b) Is it still true if $f_n \to f$ in measure.

10. Let m be the Lebesque measure on $[0,1]$ and let $f : [0,1] \to [0,\infty]$ be Lebesque integrable and $\int f \mathrm{d}m = \int f^n \mathrm{d}m$ for every positive integer n. Prove that f is a characteristic function.

11. **BONUS PROBLEM:** You may replace any of the other problems with this one:
Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $\{f_n\}$ be a sequence in $L^p(X)$ and $\{g_n\}$ be a sequence in $L^q(X)$, where $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq 1, q \geq 1$. If $\lim_{n\to\infty} f_n = f$ in $L^p(X)$ and $\lim_{n\to\infty} g_n = g$ in $L^q(X)$, is it true that $\lim_{n\to\infty} f_n g_n = fg$ in $L^1(X)$? Justify your answer.