Math 102
Foundations of Smooth Manifolds
Fall 2011
Assignment 4
Due November 4, 2011

1. Let $F : N \to M_1 \times M_2$ given by $F(p) = (f_1(p), f_2(p))$ be a smooth map. Show that for any $p \in N$ $F_* : T_pN \to T_{F(p)}M_1 \times M_2 \equiv T_{f_1(p)}N_1 \times T_{f_2(p)}N_2$ is given by $F_*(X) = (f_{1*}(X), f_{2*}(X))$.

2. Let G be a Lie group, $m : G \times G \to G$ denote multiplication and $i : G \to G$ denote inversion.

   (a) Show that $m_* : T_{(e,e)}(G \times G) \equiv T_eG \times T_eG \to T_eG$ is given by $m_*(X, Y) = X + Y$.

   (b) Let $f : G \to G$ and $h : G \to G$ be smooth functions such that $f(e) = h(e) = e$, and let $F : G \to G$ be given by $F(g) = f(g)h(g)$. Show that $F_* : T_eG \to T_eG$ is given by $F_*(X) = f_*(X) + h_*(X)$.

   (c) Show that $i_* : T_eG \to T_eG$ is given by $i_*(X) = -X$.

3. Let $F : M \to N$ be a smooth map. A point $p \in M$ is said to be a *critical point* of F if $F_* : T_pM \to T_{F(p)}N$ is not surjective; otherwise, we say that p is a *regular point*. A point $q \in N$ is said to be a regular value if each $p \in F^{-1}(q)$ is a regular value. Show that if $q \in N$ is a regular value and $S \equiv F^{-1}(q)$ is non-empty, then

   (a) S is an imbedded submanifold of dimension $\dim M - \dim N$ (**Hint:** Show that the set of regular values of F is an open subset of M.);

   (b) for each $p \in S$, the tangent space T_pS can be identified with the subspace $\ker(F_* : T_pM \to T_{F(p)}N)$.

4. Consider the determinant map $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}$.

   (a) Using the matrix entries as global coordinates on $\mathrm{GL}_n(\mathbb{R})$ show that

   $$\frac{\partial}{\partial x_{ij}} \det X = \det(X)(\hat{x}_{ij}),$$

   where $X^{-1} = (\hat{x}_{ij})$. (**Hint:** Expand det by minors along the i-th column and use Cramer's Rule.)

   (b) Conclude that for each $X \in \mathrm{GL}_n(\mathbb{R})$ the differential $\det_* : T_X \mathrm{GL}_n(\mathbb{R}) \to T_{\det(X)}\mathbb{R}$ is given by $\det_*(B) = \det(X)\,\mathrm{tr}(X^{-1}B)$, for any $B \in T_X \mathrm{GL}_n(\mathbb{R}) \equiv M_{n\times n}(\mathbb{R})$.

   (c) Conclude that $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}$ is a submersion.

   (d) Show that $\mathrm{SL}_n(\mathbb{R}) = \det^{-1}(1)$ is an imbedded submanifold of dimension $n^2 - 1$ and it is therefore a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$.

5. Boothby IV.4.4

6. Boothby IV.4.7

7. Boothby IV.5.6

8. Boothby IV.6.5