Math 102
Foundations of Smooth Manifolds
Fall 2011
Assignment 1
Due October 5, 2011

- **Part A:** Boothby Chp. III: 1.1, 1.3, 2.2, 2.3, 2.9, 3.3, 3.4 & 3.6
- **Part B:**
  1. Let V be an n-dimensional real vector space and let $\mathrm{Aut}(V)$ be the space of all linear isomorphisms $T : V \to V$. Show that $\mathrm{Aut}(V)$ is a topological manifold of dimension n^2.
  2. Let G be a topological Lie group and M a topological n-manifold. A *continuous action* of G on M is a continuous map $\Theta : G \times M \to M$ such that (1) $\Theta(e, p) = p$ for any $p \in M$; (2) $\Theta(g_1 \cdot g_2, p) = \Theta(g_1, \Theta(g_2, p))$ for any $g_1, g_2, \in G$ and $p \in M$. Sometimes we write $g \cdot p = \Theta(g, p)$. Please answer the following questions concerning continuous Lie group actions.
     (a) The standard action of $\mathrm{Aut}(V)$ on V is given by $(T, v) \mapsto T(v)$. Show that this defines a continuous group action, where V has the usual topology.
     (b) Let G be a topological Lie group and V a finite dimensional real vector space (with the usual topology). A *real representation* of G on V is a continuous group homomorphism $\rho : G \to \mathrm{Aut}(V)$. Any real representation (V, ρ) of G on V defines an action $\Theta : G \times V \to V$ given by $\Theta(g, v) = \rho(g)v$. Show that $\rho : G \to \mathrm{Aut}(V)$ is a representation if and only if Θ is a continuous group action such that $\Theta_g \equiv \Theta(g, \cdot) \in \mathrm{Aut}(V)$ for each $g \in G$.
     (c) Given a real representation (V, ρ) of a topological group G, a subspace $W \leq V$ is said to be *G-invariant* if $\rho(g)(W) \subset W$ for each $g \in G$. (V, ρ) is said to be *irreducible* if the only G-invariant subspaces of V are the trivial space and V itself. Show that if G is a finite topological Lie group, then every finite dimensional real representation (V, ρ) can be expressed as a direct sum $V = W_1 \oplus \cdots \oplus W_k$, where each W_j is G-invariant. (**Hint:** Construct a G-invariant inner product on V.)