Mathematics 101

Fall 2014

Homework 4

1. (#27, 10.3) You will demonstrate that over a noncommutative ring, the notion of rank of a module is not well-defined.

   Let $M = \prod_{n=1}^{\infty} \mathbb{Z}$ be the direct product of a countably infinite number of copies of $\mathbb{Z}$, and let $R = \mathrm{End}_{\mathbb{Z}}(M)$. Define $\varphi_1, \varphi_2 \in R$ by

   $$\varphi_1(a_1, a_2, \dots) = (a_1, a_3, a_5, \dots),$$
   $$\varphi_2(a_1, a_2, \dots) = (a_2, a_4, a_6, \dots).$$

   (a) Prove that $\{\varphi_1, \varphi_2\}$ is a basis of the left R-module R. *D&F hint:* Define maps $\psi_1, \psi_2 \in R$ by $\psi_1(a_1, a_2, \dots) = (a_1, 0, a_2, 0, \dots)$ and $\psi_2(a_1, a_2, \dots) = (0, a_1, 0, a_2, \dots)$. Verify that $\varphi_i\psi_i = 1$, $\varphi_1\psi_2 = 0 = \varphi_2\psi_1$ and $\psi_1\varphi_1 + \psi_2\varphi_2 = 1$. Use this to establish the result.

   (b) Show that $R \cong R^2$, and deduce $R^m \cong R^n$ for any $m, n \geq 1$.

2. Let R be a ring with identity, and $\{M_i\}_{i\in I}$ a collection of left R-modules. For each direction of the statements below, provide a proof or a counterexample.

   (a) $\oplus_{i\in I} M_i$ is free if and only if M_i is free for each $i \in I$.

   (b) $\oplus_{i\in I} M_i$ is projective if and only if M_i is projective for each $i \in I$.

3. Let R be a commutative ring with identity, and $S \subset R$ a multiplicatively closed set containing 1, $0 \notin S$, and let $i_R : R \to S^{-1}R$ be the ring homomorphism taking $r \mapsto r/1$. For an ideal $I \subseteq R$ define $S^{-1}I = \{i/s \mid i \in I, s \in S\}$.

   (a) Show that $S^{-1}I$ is an ideal in $S^{-1}R$ which is proper iff $I \cap S = \emptyset$. By example show that it is not the case that if $a/s \in S^{-1}I$ we must have $a \in I$.

   (b) Show that every ideal $J \subset S^{-1}R$ is of the form $S^{-1}I$ for an ideal $I \subseteq R$.

   (c) Show that there is a one-to-one correspondence between the prime ideals of $S^{-1}R$ and the prime ideals of R which are disjoint from S.

4. Let R be a commutative ring with identity, and $S \subset R$ a multiplicatively closed set containing 1, $0 \notin S$.

   (a) Show that S^{-1} is an exact functor, that is, given an exact sequence of left R-modules

   $$L \xrightarrow{\varphi} M \xrightarrow{\psi} N$$

   show that

   $$S^{-1}L \xrightarrow{S^{-1}\varphi} S^{-1}M \xrightarrow{S^{-1}\psi} S^{-1}N$$

   is an exact sequence of $S^{-1}R$-modules.

   (b) Show that for R-modules A, B, $S^{-1}(A \oplus B) \cong S^{-1}A \oplus S^{-1}B$ (as $S^{-1}R$-modules).

5. Let R be a commutative ring with identity, and $\mathfrak{P}$ a prime ideal in R (a proper ideal with the property that $ab \in \mathfrak{P}$ implies $a \in \mathfrak{P}$ or $b \in \mathfrak{P}$). With $S = R \setminus \mathfrak{P}$, we call $S^{-1}R$ the localization of R at the prime ideal $\mathfrak{P}$, usually denoted $R_{\mathfrak{P}}$.

   (a) Show that $R_{\mathfrak{P}}$ is a local ring, that is a ring containing a unique maximal ideal.

   (b) Also in this context, when M is an R-module we usually denote $S^{-1}M$ by $M_{\mathfrak{P}}$. Show the following are equivalent:

      i. $M = 0$.

      ii. $M_{\mathfrak{P}} = 0$ for all prime ideals $\mathfrak{P}$.

      iii. $M_{\mathfrak{M}} = 0$ for all maximal ideals $\mathfrak{M}$. *Hint*: To show (iii) implies (i), suppose that $M \neq 0$ and choose $m \in M$, $m \neq 0$. Let $I = \{r \in R \mid rm = 0$, the annihilator of m in R. It is a proper ideal (why?) hence contained in a maximal ideal, $\mathfrak{M}$. Now consider $m/1 \in M_{\mathfrak{M}}$.

6. Let R be a commutative ring with identity, and let $\varphi : M \to N$ be an R-linear map.

   (a) Show that the following are equivalent:

      i. φ is injective.

      ii. $\varphi_{\mathfrak{P}}$ $(= S^{-1}\varphi) : M_{\mathfrak{P}} \to N_{\mathfrak{P}}$ is injective for all prime ideals $\mathfrak{P} \subset R$.

      iii. $\varphi_{\mathfrak{M}}$ $(= S^{-1}\varphi) : M_{\mathfrak{M}} \to N_{\mathfrak{M}}$ is injective for all maximal ideals $\mathfrak{M} \subset R$.

      Remark: The analogous result is true if we replace injective by surjective.

   (b) Show that if M is a projective module, then so is $M_{\mathfrak{P}}$ for all prime ideals $\mathfrak{P}$. The converse is also true, but we need just a bit more to prove it.