Mathematics 101
Fall 2014
Homework 2

1. Irreducible and Indecomposable Modules

   (a) Show that if an R-module M is irreducible, then it is cyclic, that is $M = Rm$ for some $m \in M$. Characterize all irreducible $\mathbb{Z}$-modules.

   (b) Show that $\mathbb{Q}$ is an indecomposable $\mathbb{Z}$-module.

   (c) Let $V = k^2$ be a two dimensional vector space over a field k, and let T be a linear operator on V so that with respect to some basis, the matrix of T has the form: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Viewing V as a $k[x]$-module (induced by T), show that V is reducible, but indecomposable.

2. Let R be a ring with identity. Show that the sequence of left R-modules

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N$$

is exact if and only if for all left R-modules D, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\varphi_*} \mathrm{Hom}_R(D, M) \xrightarrow{\psi_*} \mathrm{Hom}_R(D, N)$$

is exact.

Hint: We have done the forward direction in class; for the converse, a single propitious choice of D can work, but you still need to sweat the details.

3. Let R be a ring with identity. Show that the sequence of left R-modules

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$$

is exact if and only if for all left R-modules D, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, D) \xrightarrow{\psi^*} \mathrm{Hom}_R(M, D) \xrightarrow{\varphi^*} \mathrm{Hom}_R(L, D)$$

is exact.

Hint: We have done the forward direction in class. The converse is more complicated than the covariant version; you may want to choose different modules D to establish the various conditions determining exactness of the original sequence. For example, to show ψ is surjective, let $D = N/Im(\psi)$ (the cokernel of ψ), and $\pi : N \to D$ the natural projection. Now consider $\psi^*(\pi)$ and its implications.

As a second hint, to show $Im(\varphi) \subseteq Ker(\psi)$, you need only show that $\psi \circ \varphi = 0$. Choose $D = N$ and consider the identity map $id_N \in \mathrm{Hom}_R(N, D) = \mathrm{Hom}_R(N, N)$.

4. Let R be a ring with identity. An R-module M is finitely generated if there is a finite subset $\{m_1, \ldots, m_t\}$ of M so that every element of M can be written as an R-linear combination of the m_i.

   Consider the short exact sequence of R-modules:

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$$

   (a) Show that if L and N are finitely generated, so is M.

   (b) Show that if M is finitely generated, so is N.

   (c) Show by example that if M is finitely generated, L need not be.

5. Determine the number of group homomorphisms $\mathbb{Z}_{12} \oplus \mathbb{Z}_{14} \to \mathbb{Z}_{20}$, and explicitly characterize them by specifying their action on $(\bar{1}, \bar{0})$ and $(\bar{0}, \bar{1})$.