Mathematics 101

Fall 2014

Homework 1

1. Let V be a finite dimensional vector space over a field k, and let $\varphi : V \to V$ be k-linear.
   (a) Show that there is a positive integer m so that $Im(\varphi^m) \cap ker(\varphi^m) = \{0\}$.
   (b) Now suppose that $\varphi^2 = 0$. Show that the rank of φ is at most $\dim V/2$.

2. Let V be a finite dimensional vector space over a field k, and let $\varphi : V \to V$ be k-linear, and suppose that $\varphi^2 = \varphi$, that is, φ is an idempotent map.
   (a) Show that $V = ker(\varphi) \oplus Im(\varphi)$.
   (b) Show that there is a basis of V so that the matrix with respect to this basis is diagonal all of whose entries are 0 or 1.

3. Let V be an arbitrary vector space over field k, and φ a linear operator on V. Let W be a subspace which is invariant under φ. Consequently there are induced maps, $\varphi|_W : W \to W$ and $\overline{\varphi} : V/W \to V/W$, the later defined by $\overline{\varphi}(v + W) = \varphi(v) + W$.
   (a) Show that if $\varphi|_W$ and $\overline{\varphi}$ are nonsingular (i.e., injective), then φ is nonsingular.
   (b) Show that the converse holds if V has finite dimension, and find a counterexample with V infinite dimensional.

4. Let V be a vector space over a field k, and φ a linear operator on V. Suppose that $\lambda_1, \ldots, \lambda_r$ are distinct eigenvalues of φ. For an eigenvalue λ, denote by E_λ the corresponding eigenspace, i.e., $E_\lambda = \{v \in V \mid \varphi(v) = \lambda v\}$.
   (a) Show that $\sum_{i=1}^{r} E_{\lambda_i} = \bigoplus_{i=1}^{r} E_{\lambda_i}$. Hint: It suffices to show that $E_{\lambda_1} \cap \sum_{i=2}^{r} E_{\lambda_i} = \{0\}$.
   (b) Conclude that any linear transformation on a finite dimensional vector space has at most $\dim(V)$ distinct eigenvalues.

5. Let α, β, γ be nonzero real numbers, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the subspace $W = \{(x, y, z) \mid \alpha x + \beta y + \gamma z = 0\}$. Find the matrix of T with respect to the standard ordered basis of $\mathbb{R}^3$. Hint: It would be productive to find the matrix of T with respect to a natural basis for the problem, and then use change of basis matrices to achieve the desired result.

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6. Let k be a field and $P \in GL_m(k)$.

   (a) Given a basis $\mathcal{C}$ for k^m, show there is a unique basis $\mathcal{B}$ so that $P = {}_{\mathcal{C}}[Id]_{\mathcal{B}}$.

   (b) Given a basis $\mathcal{B}$ for k^m, show there is a unique basis $\mathcal{C}$ so that $P = {}_{\mathcal{C}}[Id]_{\mathcal{B}}$.

   (c) Let A be an $m \times m$ matrix with entries from k. Show there are matrices $P, Q \in GL_m(k)$ with PAQ a diagonal matrix with zeros or ones on the diagonal.

7. Something similar is true over $\mathbb{Z}$ as well. For $A \in M_m(\mathbb{Z})$, there exist $P, Q \in GL_m(\mathbb{Z})$ so that $PAQ = \operatorname{diag}(d_1, d_2, \ldots, d_m)$ with $d_1 \mid d_2 \mid \cdots \mid d_m$. This is called the Smith Normal form of a matrix. Remember that multiplication by P and Q correspond to elementary row and column operations of the matrix. Use this observation to analyze the following situation.

   Let M be the $\mathbb{Z}$-module $\mathbb{Z}^2$ and N the submodule generated by $\left(\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 8 \\ 10 \end{smallmatrix}\right)$. The quotient module M/N is a finitely generated abelian group; indeed in this case a finite abelian group. Write it as a product of cyclic groups. Hint: Define $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $\varphi\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right)$, and $\varphi\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 8 \\ 10 \end{smallmatrix}\right)$. Then $Im(\varphi) = N$. Let $\mathcal{B} = \{e_1 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), e_2 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\}$, $\mathcal{C} = \{f_1 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), f_2 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\}$ be the standard ordered basis of $\mathbb{Z}^2$ (one for the domain, the other for the codomain). Then ${}_{\mathcal{C}}[\varphi]_{\mathcal{B}} = \left(\begin{smallmatrix} 2 & 8 \\ 4 & 10 \end{smallmatrix}\right)$. Now consider how changing the bases in domain and codomain affect $Im(\varphi)$.