Math 101 Fall 2013
MidTerm
Due Friday, October 25, 2013

INSTRUCTIONS: You are allowed to use your lecture notes and a textbook of your choice (either Lang or one of the other texts on reserve). No other resources are allowed — animate or inanimate — with the one exception that you can ask me for clarification. Monitor the web page for corrections and typos.

If you are not using LaTeX, then use one side of the paper only and start each problem on a separate page.

Unless stated otherwise, R denotes a (possibly noncommutative) ring with identity. Ideal always means two-sided ideal.

1. (12) Let R be a PID. Let $\{r_1, \dots, r_k\} \subset R \setminus \{0\}$. We say that d is a $\gcd\{r_1, \dots, r_k\}$ if $d \mid r_i$ for all i and if $c \mid r_i$ for all i, then $c \mid d$. Similarly, we say m is a $\operatorname{lcm}\{r_1, \dots, r_k\}$ if $r_i \mid m$ for all i and if $r_i \mid c$ for all i then $m \mid c$. (When it exists, we call d "the" greatest common divisor and m "the" least common multiple. We'll assume it is clear that if d and m exist, then they are unique up to associates.)

   (a) Show that $(r_1, \dots, r_k) = \gcd\{r_1, \dots, r_k\}$. In particular, gcd's always exist in PIDs.

   (b) Similarly, show that $\operatorname{lcm}\{r_1, \dots, r_k\}$ exists.

   (c) Prove that if $(a, b) = 1$ and if $a \mid bc$, then $a \mid c$.

   (d) Let M be a torsion module over R such that $M = M_1 \oplus \cdots \oplus M_k$. Let the exponent of M_i be r_i. Show that the exponent of M is $\operatorname{lcm}\{r_1, \dots, r_k\}$.

2. (10) List the possible isomorphism classes of abelian groups of order $144 = 9 \times 16$. Show both the invariant factor decomposition and the elementary divisor decomposition for each class.

3. (10) Let $V = V_1 \oplus \cdots \oplus V_r$ be a decomposition of a vector space over a field F into a direct sum of subspaces. Let β_i be a basis for each V_i. Show that $\beta = \bigcup_i \beta_i$ is a basis for V.

4. (20) Find all rational and Jordan canonical forms of a matrix A in $M_5(\mathbf{C})$ with minimal polynomial $m_A(x) = x^2(x-2)$. Be sure to give the corresponding invariants and the characteristic polynomial $c_A(x)$.

5. (20) Let $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{\pi} M'' \longrightarrow 0$ be a short exact sequence of R-modules.

(a) If M' and M'' are finitely generated, must M be finitely generated?

(b) If M is finitely generated, must M' or M'' be finitely generated?

(c) If M' and M'' are free, must M be free?

(d) If M is free, must M' or M'' be free? What if R is a PID?

6. (16) Let V be a finite-dimensional real vector space and $T \in \hom_{\mathbf{R}}(V, V)$ a linear transformation such that $T^2 = -I$. Show that the dimension of V must be even, say equal to $2r$, and that there is a basis β for V such that

$$[T]_\beta^\beta = \begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}$$

where, of course, I_r is the $r \times r$-identity matrix.

7. (12) An ideal I in a ring R is called nilpotent if $I^n = \{0\}$ for some n. (For example, consider $p\mathbf{Z}/p^k\mathbf{Z}$ in $\mathbf{Z}/p^k\mathbf{Z}$.) Show that if I is a nilpotent ideal in R and if $\phi : M \to N$ is an R-module map such that the induced map $\bar{\phi} : M/(I \cdot M) \to N/(I \cdot N)$ is surjective, then ϕ is surjective.