Math 101 Fall 2013
Homework #7
Due Friday, November 15, 2013

1. Let R be a unital subring of E. Show that $E \otimes_R R$ is isomorphic to E.

ANS: The map $(s,r) \mapsto sr$ is a R-balanced map of $E \times R$ to E. Hence there is a group homomorphism $\phi : E \otimes_R R \to E$ satisfying $s \otimes r \mapsto sr$. Since these rings are unital, ϕ is clearly surjective. But if $t = \sum_{i=1}^n s_i \otimes r_i$ is in the kernel of ϕ, then $\sum_{i=1}^n s_i r_i = 0$. But then

$$t = \sum_{i=1}^n s_i \otimes r_i = \sum_{i=1}^n s_r r_i \otimes 1 = \big(\sum_{i=1}^n s_i r_i\big) \otimes 1 = 0.$$

Hence ϕ is a group isomorphism. It then follows that the two are isomorphic as left E-modules, right R-modules or even (E,R)-bimodules.
Remark: If E is commutative or if R is in the center of E, then both E and R are R-algebras. In particular, $E \otimes_R R$ has a ring structure and ϕ is easily seen to be a ring isomorphism.

2. Show that $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathbf{Q} \otimes_{\mathbf{Q}} \mathbf{Q}$ are isomorphic. (Show both are vector spaces over $\mathbf{Q}$ of dimension one.)

ANS: Both are left $\mathbf{Q}$-modules and hence vector spaces over $\mathbf{Q}$. Hence it suffices to see that they have the same dimension. But $\mathbf{Q} \otimes_{\mathbf{Q}} \mathbf{Q} \cong \mathbf{Q}$, so it clearly has dimension 1. But if $\frac{a}{b} \otimes \frac{c}{d} \in \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q}$, then

$$\frac{a}{b} \otimes \frac{c}{d} = \frac{bd}{bd}\Big(\frac{a}{b} \otimes \frac{c}{d}\Big) = \frac{1}{bd}(a \otimes c) = \frac{ac}{bd}(1 \otimes 1).$$

Hence $\{1 \otimes 1\}$ is a spanning set for $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q}$. This suffices.

3. Show that as left $\mathbf{R}$-modules, $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ and $\mathbf{C} \otimes_{\mathbf{C}} \mathbf{C}$ are not isomorphic.

ANS: $\mathbf{C} \otimes_{\mathbf{C}} \mathbf{C} \cong \mathbf{C}$, which is a two-dimensional real vector space. On the other hand, as a real vector space, $C \cong \mathbf{R}^2$. Hence $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{R}^2 \otimes_{\mathbf{R}} \mathbf{R}^2$ which is isomorphic to $\mathbf{R}^4$, which is a 4-dimensional real vector space.

4. Recall that for R-modules, we write $\bigoplus_{i \in I} M_i$ in place of the coproduct $\coprod_{i \in I} M_i$. Then show that tensor products commute with direct sums. That is, show that

$$N \otimes_R \bigoplus_{i \in I} M_i \cong \bigoplus_{i \in I} N \otimes_R M_i,$$

and that an isomorphism is given by $n \otimes (m_i) \mapsto (n \otimes m_i)$. (I suggest using the universal property of the tensor product. What assumptions are you making about R, N and the M_i?)

ANS: To begin with, we assume that N is a right N-module and that each M_i is a left R-module. Let $i : N \times \bigoplus_i M_i \to N \otimes_R \bigoplus_i M_i$ be the universal R-balanced map $i(n,(m_i)) = n \otimes (m_i)$. I claim $j : N \times \bigoplus_i M_i \to \bigoplus_i N \otimes_R M_i$ is also universal, where $j(n,(m_i)) = (n \otimes m_i)$. Let $f : N \times \bigoplus M_i \to A$ be an R-balanced map into an abelian group A. Let $i_k : M_k \to \bigoplus_i M_i$ be the natural injection. Then $j \circ i_k$ is an R-balanced map of $N \times M_{i_k}$ into A. Hence there is a unique group homomorphism $j_k : N \otimes M_{i_k} \to A$ such that

$$\begin{tikzcd}
 & & N \otimes_R M_{i_k} \arrow[d, "j_k"] \\
N \times M_{i_k} \arrow[urr, "i"] \arrow[r, "1\times i_k"'] & N \times \bigoplus_i M_i \arrow[r, "f"'] & A
\end{tikzcd}$$

commutes. Then we can define $\bar{f} : \bigoplus_i N \otimes_R M_i \to A$ by $\bar{f}((n \otimes m_i)) = \sum_i j_i(n \otimes m_i)$. (This is well defined only because only finitely many terms are nonzero.) But since f is R-balanced, and hence additive in its second variable, and since $(m_i) = \sum_i m_i \cdot \epsilon_i$, we see that $f(n,(m_i)) = \sum_i f(n, m_i \cdot \epsilon_i) = \sum j_i(n, m_i)$. (With the last equality due to the commutativity of the above diagram. That is, the diagram

$$\begin{tikzcd}
 & \bigoplus N \otimes_R M_i \arrow[d, "\bar{f}"] \\
N \times \bigoplus M_i \arrow[ur, "j"] \arrow[r, "f"'] & A
\end{tikzcd}$$

commutes.

Since both i and j are universal, the result follows easily: we get unique group homomorphisms $\bar{i}$ and $\bar{j}$ such that

$$\begin{tikzcd}
 & \bigoplus N \otimes_R M_i \arrow[dd, shift left, "\bar{j}"] \\
N \times \bigoplus M_i \arrow[ur, "j"] \arrow[dr, "i"'] & \\
 & N \otimes_R \bigoplus M_i \arrow[uu, shift left, "\bar{i}"]
\end{tikzcd}$$

commutes. Uniqueness forces $\bar{i}$ and $\bar{j}$ to be inverses of one another. In particular, $\bar{i}(n \otimes (m_i)) = j(n,(m_i)) = (n \otimes m_i)$ as required.

Note that if N is a (E,R)-bimodule, then $\bar{i}$ is easily seen to be an E-module map. Or if each M_i is a (R,P)-bimodule, then $\bar{i}$ is a right P-module map.

5. Let A be a finite abelian group of order $p^\alpha m$ with $p \nmid m$. Prove that $\mathbf{Z}_{p^\alpha} \otimes_{\mathbf{Z}} A$ is isomorphic to the p-Sylow subgroup of A.

ANS: Since all the Sylow subgroups of an abelain group are normal and have trivial intersection with one another, A is the internal direct sum $\bigoplus_i A(p_i)$ where $A(p_i)$ is the p_i-Sylow subgroup. We can assume $p_1 = p$. Recall that $\mathbf{Z}_n \otimes_{\mathbf{Z}} \mathbf{Z}_m = \{0\}$ if $(n, m) = 1$. Hence $Z_{p^\alpha} \otimes_Z A \cong \mathbf{Z}_{p^\alpha} \otimes A(p)$. On the other hand, $A(p) \cong \bigoplus_k \mathbf{Z}_{p^{n_k}}$ with $\sum n_k = \alpha$. In particular, $p^\alpha \geq p^{n_k}$ for all k. But then

$$\mathbf{Z}_{p^\alpha} \otimes_{\mathbf{Z}} \bigoplus \mathbf{Z}_{p^{n_k}} = \bigoplus \mathbf{Z}_{p^{n_k}} = A(p).$$

6. Recall that if S is a multiplicative subset of a commutative ring R and M is an R-module, then we can form the fraction module $S^{-1}M$. Because both share the same universal property, we also observed that $\frac{m}{s} \mapsto \frac{1}{1} \otimes m$ induces an isomorphism of $S^{-1}M$ onto $S^{-1}R \otimes_R M$. Except for part (a), we'll take $R = \mathbf{Z}$ and $S^{-1}R = \mathbf{Q}$ in this problem. But you might want to think about generalizations of parts (b) and (c).

(a) Show that $\frac{1}{s} \otimes m$ is zero in $S^{-1}M \otimes_R M$ if and only if there is a $s' \in S$ such that $s' \cdot m = 0$. ("Use the isomorphism Luke.")

(b) Let A be an abelian group. Show that $\mathbf{Q} \otimes_{\mathbf{Z}} A = \{0\}$ if and only if A is torsion.

(c) Recall that if $\phi : M' \to M$ is an R-module map, then we get a homomorphism $1 \otimes \phi : N \otimes_R M' \to N \otimes_R M$ for any right R-module N characterized by $\phi(n \otimes m') = n \otimes \phi(m')$. Show that if

$$1 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 1$$

is a short exact sequence of abelian groups, then

$$1 \longrightarrow \mathbf{Q} \otimes_{\mathbf{Z}} A \xrightarrow{1 \otimes i} \mathbf{Q} \otimes_{\mathbf{Z}} B \xrightarrow{1 \otimes j} \mathbf{Q} \otimes_{\mathbf{Z}} C \longrightarrow 1$$

is a short exact sequence of vector spaces. (One says that $Q \otimes_{\mathbf{Z}} \_$ is exact. And Luke, don't forget ....)

(d) Is it always true that if M' is a submodule of M then $N \otimes_R M'$ is a submodule of $N \otimes_R M$. (That is, if $\iota : M' \to M$ is the inclusion, is $1 \otimes \iota$ necessarily injective?)

ANS: (a) In view of the isomorphism above, $\frac{1}{s} \otimes m = 0$ if and only if $\frac{s}{m} = 0$ in $S^{-1}M$. But that means $\frac{s}{m} = \frac{0}{1}$. But that happens only when there is a $s' \in S$ such that $s' \cdot m = 0$.

(b) If $a \in A$ is not a torsion element, then $n \cdot a \neq 0$ for all $n \in \mathbf{Z} \setminus \{0\}$. Hence $1 \otimes a \neq 0$ by part (a). Hence if A is not all torsion, $\mathbf{Q} \otimes_{\mathbf{Z}} A \neq \{0\}$.

On the other hand, if A is torsion, then given $a \in A$, there is a $n \in Z$ such that $n \cdot a = 0$. Then $q \otimes a = \frac{1}{n} q \otimes n \cdot a = 0$. That is $\mathbf{Q} \otimes_{\mathbf{Z}} A = \{0\}$.

(c) Note that if $t = \sum q_i \otimes a_i \in \mathbf{Q} \otimes A$, then there are integers m_i and n_i such that

$$t = \sum_{i=1}^{k} \frac{m_i}{n_i} \otimes a_i = \sum_{i=1}^{k} \frac{1}{n_1 \cdots n_k} \otimes \frac{n_1 \cdots n_k}{n_i} m_i \cdot a_i = \frac{1}{n_1 \cdots n_k} \otimes \big(\sum_i \frac{n_1 \cdots n_k}{n_i} m_i \cdot a_i\big).$$

But this just says every element in $\mathbf{Q} \otimes_{\mathbf{Z}} A$ can be written in the form $\frac{1}{s} \otimes a$ for some integer s and some $a \in A$. But if $\frac{1}{s} \otimes a \in \ker 1 \otimes i$, then $\frac{1}{s} \otimes i(a) = 0$ in $\mathbf{Q} \otimes_{\mathbf{Z}} B$. But then part (a) implies that $i(a) = 0$. But i injective forces $a = 0$. Thus $1 \otimes i$ is injective.

Since $1 \otimes j$ is clearly surjective, we only have to worry about exactness in the middle. Since $(1 \otimes j) \circ (1 \otimes i)$ is clearly zero, it is enough to see that the kernel of $1 \otimes j$ is in the image of $1 \otimes i$. But if $t \in \ker 1 \otimes j$, then by the above we can assume $t = \frac{1}{s} \otimes b$ with $j(b) = 0$. But the there is an a such that $i(a) = b$, and $(1 \otimes i)(\frac{1}{s} \otimes a) = \frac{1}{s} \otimes b$.

(d) See Lecture notes.

7. In question 4, we observed the tensor products commute with direct sums. Do they commute with direct products? Let P be the set of primes in $\mathbf{Z}$ and consider $M = \prod_{p \in P} \mathbf{Z}_p$ and our old friend $\mathbf{Q} \otimes_{\mathbf{Z}} \_$.

ANS: We just need to observe that $\mathbf{Q} \otimes_{\mathbf{Z}} M \neq \{0\}$.

Note that the element $x := (\bar{1}, \bar{1}, \bar{1}, \dots)$ has infinite order in M. Hence it generates a subgroup $\langle x \rangle$ isomorphic to $\mathbf{Z}$. Since, as we now say, $\mathbf{Q}$ is flat, we have an injection of $\mathbf{Q} \cong \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}$ into $\mathbf{Q} \otimes_{\mathbf{Z}} M$. In particular, $M \neq \{0\}$.

8. If R is a unital subring of E, then we formally have two meanings for $E \otimes_R M$ for an R-module M: we first "extended the scalars from R to E", but we could also consider E as a (E, R)-bimodule and form the (general) tensor product. Explain why these are the same thing.

ANS: Originally, we defined $E \otimes_R M$ so that $\bar{i} : M \to E \otimes_R M$ was universal as a R-module map of M into an E-module L. Let's show that the full-fledged tensor product $E \otimes_R M$ has the right universal property. But if $f : M \to L$ is an R-module map into an E-module L, then $g(s, m) = sf(m)$ is an R-balanced map into L. Hence there is a E-module map $\bar{g} : E \otimes_R M$ to L such that $\bar{g}(s \otimes m) = sf(m)$. But then $\bar{f}(m) := \bar{g}(1, m)$ does the old job. Hence universal nonsense says they are the same.

9. Let $F(S)$ be a free R-module with basis S and let M be an R-module. Show that every element $\mathfrak{t}$ of $F(S) \otimes_R M$ has a *unique* representation in the form

$$\mathfrak{t} = \sum_{s \in S} s \otimes m_s, \tag{‡}$$

where only finitely many m_s are nonzero. In particular, if $\mathfrak{t} = 0$, then all the m_s are zero. (If we specialize to the case where S is finite and $F(S) = R^n$, every element of $R^n \otimes_R M$ has a unique representative of the form

$$\sum_{i=1}^{n} e_i \otimes m_i$$

where e_i is the usual basis vector and $m_i \in M$. We also note that it is critical that S be a basis. The set $S = \{s_1, s_2\} = \{(2,0), (0,2)\}$ is $\mathbf{Z}$-linearly independent in $\mathbf{Z}^2$, but $s_1 \otimes 1 + s_2 \otimes 1 = 0$ in $\mathbf{Z}^2 \otimes_{\mathbf{Z}} \mathbf{Z}_2$.)

ANS: Note that every element x in $F(S)$ can be written uniquely as $\sum_{s\in S} s \cdot r_s$ where all by finitely many r_s are zero. Thus $x \otimes m = \sum s \otimes r_s \cdot m$. It now follows that every element $\mathfrak{t}$ can be written in the form (‡). The map $f_{s_0} : F(S) \to R$ given by sending $\sum_{s\in S} s \cdot r_s$ to r_{s_0} is an R-module map. Similary the map $(x, m) \mapsto f_{s_0}(x) \cdot m$ is an R-balanced map from $F(S) \times M$ to M. Hence we get a map $S_{s_0} : F(S) \otimes_R M \to M$ that takes $\mathfrak{t}$ above to m_{s_0}. This implies uniqueness. Thus if $\mathfrak{t} = 0$, then $m_s = 0$ for all s.