Math 101 Fall 2013
Homework #4
Due Wednesday October 16, 2013

1. Let $R = \mathbf{Q}[x]$ and let V be the 2-dimensional rational vector space $\mathbf{Q}^2$ and let $T : V \to V$ be given by $Tv = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)v$. View V as a R-module in the usual way $p(x) \cdot v = p(T)v$. Let $u = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Find $|u|$ and $|v|$.

2. Let M be a module over a PID. Suppose that x and y are torsion elements in M with orders r and s, respectively. If $(r, s) = 1$, then show that the order of $x + y$ is rs.

3. Show that a ring R is Noetherian if and only if every ideal in R is finitely generated. In this problem R is any ring. Ideal means two-sided ideal, and Noetherian means every ascending sequence of ideals is eventually constant.

4. Suppose that R is a PID. The aim of this problem is to prove the special case of Hilbert's Theorem which says that $R[x]$ is a Noetherian ring. Let I be a nonzero ideal in $R[x]$. By question 3, it will suffice to show that I is finitely generated. For each $n \geq 0$, let A_n be the union of the zero element and all elements of R which occur as leading coefficients of polynomials of degree n in I. (Thus $a_n \in A_n \setminus \{0\}$ if and only if there is a $p(x) \in I$ of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$.)

(a) Show that each A_n is an ideal in R and that $A_n \subset A_{n+1}$ for all $n \geq 0$.

(b) Conclude that there is an r such that $A_n = A_r$ for all $r \geq n$.

(c) Since R is a PID, we have $A_n = (a_n)$ and there is a degree n polynomial $p_n(x)$ in I with leading coefficient a_n. Let J be the ideal in $R[x]$ generated by $\{p_0(x), \ldots, p_r(x)\}$. Then given any polynomial $f(x)$ of degree d in I, that there a polynomial $g(x) \in J$ such that the degree of $f(x) - g(x)$ is strictly less than d.

(d) Conclude that $I = J$. Hence I is finitely generated.

5. Let M be a module over a PID R. Suppose that $m \in M$ has order r. If $s \in R$, show that $\langle m \rangle[s] = \langle \frac{r}{(r,s)} \cdot m \rangle \cong R/(r, s)$. (Here "$(r, s)$" is used both to designate the ideal generated by r and s as well as the generator of that ideal.)

6. Let F be a field and give $R = \prod_{n=1}^{\infty} F$ the obvious ring structure. Let $I = \{ (x_i) \in R : x_i = 0 \text{ for all by finitely many } i \}$.

(a) Observe that I is an ideal in R. Hence I is an R-module.

(b) Show that I is not a finitely generated R-module.

(c) Conclude that submodules of finitely generated modules need not be finitely generated.

7. Here and elsewhere, V_T is the $F[x]$-module corresponding to a finite-dimensional F-vector space V and linear map $T : V \to V$. Suppose that $W = \langle v \rangle$ is a cyclic submodule of V_T of order $f(x)$ for $f \in F[x]$ with $\deg f = k > 0$. Show that $\{v, Tv, T^2 v, \dots, T^{k-1} v\}$ is a (vector space) basis for W.