Math 101 Fall 2013
Homework #2
Due 2 October 2013

1. Let $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{\pi} M'' \longrightarrow 0$ be a short exact sequence of R-modules. Show that if i has a retraction $r : M \to M'$, then $M \cong M' \oplus M''$.

2. Let M be an R-module and let $S \subset M$ be a subset. Show that there is a smallest submodule, $\langle S \rangle$, of M containing S. We say that $\langle S \rangle$ is the submodule generated by S. Of course, if $\langle S \rangle = M$, then we say that S generates M. Now let S be any set and $F(S)$ together with $i : S \to F(S)$ a free module on S. Show that $F(S)$ is generated by $i(S)$.

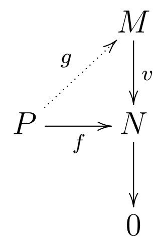
3. Give an example of a group G with subgroups H and K such that $HK = \{ hk : h \in H \text{ and } k \in K \}$ is not a subgroup of G. (Groups start to get interesting at $G = S_3$.)

4. Let $\mathbf{Q}$ be the additive group of rationals. Show that $\mathbf{Q}$ is indecomposable as a $\mathbf{Z}$-module: that is, show that it is not possible to write $\mathbf{Q} \cong A \oplus B$ for $\mathbf{Z}$-modules A and B. Conclude that $\mathbf{Q}$ is not a free $\mathbf{Z}$-module.

5. Let $\mathbf{Q}^\times$ be the multiplicative group of nonzero rational numbers. Show that as a $\mathbf{Z}$-module, $\mathbf{Q}^\times \cong \left(\coprod_{i=1}^{\infty} \mathbf{Z}\right) \oplus \mathbf{Z}_2$. (First write $\mathbf{Q}^\times \cong H \oplus K$ where $H = \{ q \in \mathbf{Q}^\times : q > 0 \}$. Let $\{p_i\}$ be the set of primes in $\mathbf{N}$ and define $\phi_i : \mathbf{Z} \to H$ by $\phi_i(k) = p_i^k$.)

6. In lecture, we proved that if R is a *commutative* ring, then $R^n \cong R^m$ as R-modules if and only if $n = m$. If R is not commutative, this is no longer true. Show that if V is a (countably) infinite dimensional k-vector space and if $R = \mathrm{End}_k(V) = \hom_k(V, V)$, then $R \cong R \oplus R$ (as R-modules). (You might want to start by observing that $\hom_k(V, V)$ has a nice ring structure.)

7. An R-module P is called *projective* if whenever we have an R-module epimorphism $v : M \to N$ and R-module map $f : F \to N$ there is an R-module map g lifting f in the sense that the diagram



commutes. (Note that g is not required to be unique.) Show that P is projective if and only if P is a direct summand of a free R-module (i.e., there is an R-module Q such that $P \oplus Q$ is free).

8. Recall that an ideal in a ring R is called *prime* if $ab \in I$ implies that either $a \in I$ and $b \in I$. Show that in a commutative ring R and ideal I is prime if and only if R/I is an integral domain.

9. Suppose that p is a prime and that $P = p\mathbf{Z}$ is the corresponding prime ideal in $\mathbf{Z}$. Then $\mathbf{Z}_P$ is the ring $S^{-1}\mathbf{Z}$ for $S = \mathbf{Z} \setminus p\mathbf{Z}$. Show that $\mathbf{Z}_P$ can be realized as the subring of $\mathbf{Q}$ given by $\{\, \frac{a}{b} : a, b \in \mathbf{Z},\, b \neq 0 \text{ and } p \nmid b \,\}$. Show $p = \frac{p}{1}$ is prime in Z_P and that every element of $\mathbf{Z}_P$ is of the form $p^\nu u$ for u a unit in $\mathbf{Z}_P$.